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1

On Thomae's Criterion

BY

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The object of the present paper is to examine Thomae's criterion* with a view to find how far it is helpful in the solution of the problem studied by me in my first paper,† "On the fundamental theorem of the Integral Calculus." Unless the contrary is explicitly stated, throughout this paper by an integral is to be understood Riemann's integral.

1. Let $f(t)$ be any function integrable in the interval $(0; x)$ and let $J(t)$ be a function whose differential coefficient is $\{f(t)/t\}$.

Then, considering x to be greater than 0,

$$\int_0^x f(t)dt = \lim_{\epsilon=+0} \int_\epsilon^x f(t)dt.$$

Now

$$\int_\epsilon^x f(t)dt = \int_\epsilon^x t.\frac{f(t)}{t}\,dt$$

$$= \Big[t.\, J(t) \Big]_\epsilon^x - \int_\epsilon^x J(t)\, dt$$

(by integration by parts)

Thus

$$\int_\epsilon^x f(t)\, dt = x\, J(x) - \epsilon J(\epsilon) - \int_\epsilon^x J(t)\, dt \qquad \ldots \quad \text{(A)}$$

* See *Gött. Nachrichten* for 1893, pp. 696-700; also this *Bulletin*, Vol. XVII, pp. 113-114.

† This *Bulletin*, Vol. XVI.

2. Let $F(x)$, as usual, stand for

$$\int_0^x f(t)\,dt.$$

Then, provided that $\epsilon\, J(\epsilon)$ tends to 0 and $J(t)$ is integrable in the nterval $(0, x)$, by making ϵ tend to zero we deduce from (A) that

$$F(x) = x\,J(x) - \int_0^x J(t)\,dt \qquad \ldots \quad (B)$$

Therefore

$$F'(+0) = \lim_{x=+0} J(x) - \lim_{x=+0} \frac{1}{x}\int_0^x J(t)\,dt$$

Similarly

$$F'(-0) = \lim_{x=-0} J(x) - \lim_{x=-0} \frac{1}{x}\int_0^x J(t)\,dt$$

Hence follows Thomae's criterion, *viz.*, that $F'(0)$ exists and equals zero provided that $J(x)$ tends to zero as x tends to 0.

3. The condition that $J(x)$ should tend to zero is *not* necessary for the existence of $F'(0)$. For, consider the function $f(t)$ for which

$$J(t) = \cos\left\{\left(\log\frac{1}{t^2}\right)^{\frac{1}{2}}\right\}.$$

Then

$$\frac{f(t)}{t} = J'(t) = \frac{1}{t\left(\log\frac{1}{t^2}\right)^{\frac{1}{2}}}\cdot\sin\left\{\left(\log\frac{1}{t^2}\right)^{\frac{1}{2}}\right\}$$

and

$$f(t) = \frac{1}{\left(\log\frac{1}{t^2}\right)^{\frac{1}{2}}}\cdot\sin\left\{\left(\log\frac{1}{t^2}\right)^{\frac{1}{2}}\right\}.$$

Therefore $f(t)$ is a continuous function at $t=0$ and, consequently, $F'(0)$ exists and equals 0.

4. Generally, it holds that, if $J(t)=\cos\psi(t)$, where $\psi(t)$ is a monotone function in the neighbourhood of $t=0$ and $\psi \succ 1$, $F'(0)$ exists and equals 0, if

$$\psi(t) \prec \log\frac{1}{t^2};$$

and does not exist, if

$$\psi(t) \succeq \log\frac{1}{t^2}.$$

Proof :—

$$J'(t) = -\psi'(t)\sin\psi(t),$$

therefore

$$f(t) = -t\psi'(t)\sin\psi(t).$$

Case (*i*) Let $\psi(t) \prec \log\frac{1}{t^2}$;

then $$\psi'(t) \prec \frac{1}{t}$$

i.e. $$t\psi'(t) \prec 1$$

and, consequently, $f(t)$ is continuous at $t=0$ and $F'(0)=0$.

Case (*ii*) Let $\psi(t) \succ \log\frac{1}{t^2}$.

Then (B) of §. 2 holds. Now, as proved in my first paper,

$$\int_0^\infty J(t)\,dt$$

has a differential co-efficient at $x=0$, which equals 0; and, therefore,

$$\operatorname*{Lim}_{x=0}\frac{F(x)}{x} = \operatorname*{Lim}_{x=0} J(x)$$

which is non-existent.

Case (*iii*). Let $\psi(t) \sim \log\frac{1}{t^2}$.

Then (B) holds and using the results of my first paper it is evident that $F'(0)$ is non-existent.

5. Let

$$J(t)=\chi(t)\cos\psi(t), \text{ where } \chi(t) \succ 1.$$

Also let $\chi \prec \frac{1}{t}$, then (B) holds. A detailed treatment of this case will be given in another paper. The following examples are intended to illustrate certain peculiarities :—

Ex. 1. Let $J(t) = \left(\log\frac{1}{t^2}\right)^{\frac{1}{3}} \cos\left\{\left(\log\frac{1}{t^2}\right)^{\frac{1}{3}}\right\}$; then $F'(0)$ is existent.

Ex. 2. Let $J(t) = \left(\log\frac{1}{t^2}\right)^{\frac{1}{2}} \cos\left\{\left(\log\frac{1}{t^2}\right)^{\frac{1}{2}}\right\}$; then $F'(0)$ is non-existent.

Ex. 3. Let $J(t) = \left(\log\frac{1}{t^2}\right) \cos\left\{\log\frac{1}{t^2}\right\}$; then $F'(0)$ is non-existent.

6. It may be concluded that, although Thomae's criterion is valid even for Lebesgue's integrals, it is essentially an insufficient guide in most cases for testing the existence or non-existence of $F'(0)$.

ERRATA

In §. 5 of the first paper read

has not	for	*has*	in	Ex. 1 and Ex. 4.
p	for	*zero*	in	Ex. 2 ;
at	for	*zero at*	in	Ex. 4.

Bull. Cal. Math. Soc., Vol. XVIII, No. 1 (1927).

2

Āryabhaṭa, the Author of the "Gaṇita"

BY

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The Gaṇita and the Āryabhaṭīya

The *Gaṇita* is the title of a chapter of the extant *Āryabhaṭīya.** This book is divided principally into two parts. The first part contains 10 verses in the Gīti metre. The second part consists of three chapters, entitled the *Gaṇita*, the *Kāla-kriyā* and the *Gola*, containing respectively 33, 25 and 50, altogether 108, verses in the Āryā metre. There are also three other verses in the same metre, two being introductory to the whole book and the third forms an adjunct to the first part, being put at its end. From the total number of verses as well as from the kind of metre used, the two parts are sometimes named separately as the *Daśagītikā* and the *Āryāṣṭaśata*. These two names are found for the first time in the *Brāhma-sphuṭa-siddhānta* of Brahmagupta.† The *Āryabhaṭīya* is the name given by its author to the whole book. This book has sometimes been called the *First-Ārya-siddhānta*, or the *Laghu-Ārya-siddhānta* by the later Indian mathematicians in order to distinguish it from another *Ārya-siddhānta* which has consequently been called the *Second-Ārya-siddhānta*, the *Mahā-Ārya-siddhānta* or simply the *Mahā-siddhānta*.‡ The two authors are namesakes; but they lived in ages separated by several centuries. In accordance with the accounts given in the *Āryabhaṭīya*, it has now been generally accepted that its author, Āryabhaṭa, was born in 476 A.D., at Kusumapura (near Patna).§

* *Āryabhaṭīya*, ed. Kern, Leiden (1874).

† *Brāhma-sphuṭa-siddhānta*, ed. Sudhakara Dvivedī, Benares (1902), ch. xi, verse 8.

‡ *Mahā-siddhānta*, ed. Sudhakara Dvivedi, Benares (1910).

§ For an uncertainty about the place of birth see p. 7 footnote.

Nothing is known definitely about the age of the second Āryabhaṭa except the fact that he was anterior to the celebrated Bhāskara (b. 1114 A.D.) who has referred to him.* In the opinion of Sankar Balkrisna Dikshit † and Sewell ‡ he lived about 950 A.D., whereas according to Sudhakara Dvivedi § he lived in the period 991-1114 A.D. Bentley, followed also by Bhau Daji, mistook him as belonging to the fourteenth century.||

Conjecture of Kaye: authorship of the Gaṇita

Contrary to the opinions of such distinguished savants as Bhau Daji,¶ Kern,** Weber,†† Rodet,‡‡ Thibaut,§§ Sankar Balkrisna Dikshit,|||| Sudhakara Dvivedi¶¶ and Fleet,*** Kaye ††† conjectures that the *Gaṇita* is the work of a writer of the 10th century whom he identifies with the "Āryabhaṭa of Kusumapura" of Al-Biruni. ‡‡‡

* Bhāskara, *Siddhānta Śiromaṇi*, ed. Bapu Deva Sastri and L. Wilkinson, Calcutta (1861); *Grahagaṇita*, *Spaṣṭādhikāra*, verse 65, (*Vāsanābhāṣya*).

† Sankar Balkrisna Dikshit, *History of Indian Astronomy*, Poona (1896), p. 230.

‡ R. Sewell, *The Siddhāntas and the Indian Calender*, Calcutta (1924), Pref, p. ix.

§ *Mahā-siddhānta*, *loc. cit.*, Contents, pp. 22-23.

|| J. Bentley, *Historical View of the Hindoo Astronomy* (1825), p. 128. For reference to Bhau Daji *vide* infra.

¶ Bhau Daji, "Brief Notes on the Age and Authenticity of the works of Āryabhaṭa, etc," *Journ. Roy. Asiat. Soc.* (1865), p. 392.

** *Vide Bṛhat Saṃhitā*, ed. Kern, Calcutta (1865), Pref., p. 59.

†† A. Weber, *History of Indian Literature*, trans. Mann & Zachariae, London (1878), pp. 257-8.

‡‡ L. Rodet, "Leçons de calcul d'Āryabhaṭa," *Journ. Asiat.* (1878), p. 373.

§§ G. Thibaut and Sudhakara Dvivedi, *Pañca-siddhāntikā*, Introduction, p. lvii.

|||| Sankar Balkrisna Dikshit, *loc cit.*, p. 190

¶¶ Sudhakara Dvivedi; *Gaṇaka Taraṅginī*, Benares (1892), p. 2.

*** J. F. Fleet, "Āryabhaṭa's system of expressing numbers," *Journ. Roy. Asiat. Soc.* (1911), p. 110.

††† G. R. Kaye, "Notes on Indian Mathematics, No 2—Āryabhaṭa," *Journ. Asiat. Soc. Beng.* (1908), p. 111; "The two Āryabhaṭas," *Bibl. Math.* (1910) xiii; *Indian Mathematics*, Calcutta (1915), p. 11. The second paper is not available to me; I have seen an abstract of it in the *Jahrbuch ü. d. Fortschritte d. Math.* (1909), p. 7.

‡‡‡ *Al-Biruni's India*, ed. E. C. Sachau, vols. I & II (1910).

The rest of the *Āryabhaṭīya* has been attributed by him to the same Āryabhaṭa to whom has it been done by all other writers. He further presumes his new author to be identical with the author of the *Mahā-siddhānta.* The grounds on which Kaye has based his conjecture may be summed up as folllow :—

(1) Al-Biruni has referred to two Āryabhaṭas, the elder Āryabhaṭa and Āryabhaṭa of Kusumapura ; the latter, a follower of the former.

(2) In the *Mahā-siddhānta* also we find its author Āryabhaṭa expressing his indebtedness to one Vṛddha (elder) Āryabhaṭa.

(3) It has been stated in the opening verse of the *Gaṇita* that it was written by Āryabhaṭa at Kusumapura.*

(4) Though Āryabhaṭa has been quoted copiously by Varāha-mihira, Brahmagupta, Bhaṭṭotpala and Bhāskara, none has drawn any quotation from the *Gaṇita.* The earliest writer to quote directly from the *Gaṇita* is Caturveda Prithuduka Swami.†

(5) Caturveda Prithudaka Swami lived in the latter part of the 10th century A.D.

Objections against Kaye's conjecture

Against this conjecture of Kaye, there are several serious objections, to some of which attention has been drawn elsewhere.‡ It has been shown there that Kaye's conjecture is unwarranted and also without any foundation for the following chief reasons amongst others :—

(1) None of the quotations attributed by Al-Biruni to his Āryabhaṭa of Kusumapura is from the *Gaṇita* whereas

* The original is

"आर्यभटस्त्विह निगदति कुसुमपुरेऽभ्यर्चितं ज्ञानम्"

It can be interpreted in two ways: the first to mean that Āryabhaṭa wrote his book at Kusumapura; the second to mean that Āryabhaṭa wrote in his book "the knowledge acquired at, or cultivated in the school of, Kusumapura."

If the second interpretation be accepted, the place of birth of Āryabhaṭa becomes uncertain.

† *cf.* Kaye, Āryabhaṭa, *loc. cit.*, p. 118.

‡ *Vide* Bibhutibhusan Datta, "Two Āryabhaṭas of Al-Biruni," *Bull. Cal. Math. Soc.*, Vol 17 (1926), p. 59.

almost all of them are found to have been drawn more or less freely from the other parts of the *Āryabhaṭīya*.

(2) There is nothing common, on the contrary, there are many things contradictory between *Āryabhaṭiya* and the *Mahā-siddhānta*. So it is very doubtful whether the author of the latter had in mind the author of the former while expressing his indebtedness to Vṛddha Āryabhaṭa.

(3) There is nothing in the *Mahā-siddhānta*, nor anything has been so far found elsewhere to hint that its author had at any time anything to do with Kusumapura.

(4) Brahmagupta has stated the *Gaṇita* to be a section of the work of Āryabhaṭa criticised by him.*

(5) According to the accounts given by Al-Biruni, his Āryabhaṭa of Kusumapura lived before the eighth century A.D.; he might have even been anterior to Brahmagupta (b. 598 A.D.).

(6) Neither did Al-Biruni himself, nor did any of his Indian informants see the original works of the either of his two Āryabhaṭas. All his informations about them were drawn from various secondary sources, Indian as well as Arabian, about the faithfulness of some of which he had occasions to question.

(7) The quotations attributed by Al-Biruni to either of his Āryabhaṭas have, save a solitary exception, more or less close parallels in the extant *Āryabhaṭīya*. It is learnt on the authority of Bhāskara that the excepted instance embodies the opinion of the Purāṇas. So it is very likely an instance of false identification by Al-Biruni.

From all these facts we were led to the conclusion that Al-Biruni did make some confusion. *Either*, he made confusion about their works and then quotations emanating from one prime source were assumed and referred to by him as being from the works of two writers. *Or*, he made confusion about their persons and then represented the one and the same individual as two different persons of the same name. This latter opinion is also shared by Sankar Balkrisna Dikshit.†

The object of the present paper is to examine more closely the uncertainty introduced by Kaye about the authorship of the *Gaṇita* and

* *Vide infra*, p. 16.

† Sankar Balkrisna Dikshit, *loc. cit.*, p. 310, footnote.

thus to establish the validity of the texts attributed to Āryabhaṭa (the elder). For unfotunately this conjecture of Kaye has got some adherents here and there, especially amongst those who have very little, or almost no, opportunity to acquire first hand knowledge of Indian mathematics on account of the difficulty of the Sanskrit language as well as of the paucity of printed materials. It is found that Kaye is utterly wrong and that the *Gaṇita* is the work of Āryabhaṭa referred to and criticised by Varāhamihira, Brahmagupta and others.

Misrepresentation by Kaye: Contradicted

Kaye remarks: "Neither does Bhāskara refer to Āryabhaṭa anywhere in his mathematical works, although Colebrooke (p. iv) says: 'He repeatedly adverts to preceding writers, and refers to them in general terms, where his commentators understand him to allude to Āryabhaṭa, to Brahmagupta, to the latter's scholiast Chaturveda Prithudaka Swami and to other writers above mentioned'."* There is a tissue of misrepresentation throughout the whole passage. Kaye would impress upon his readers that Bhāskara has referred to many writers preceding him but not to Āryabhaṭa. He has called upon the authority of Colebrooke—with whom he is hardly in love—to come to his help in this matter. It is unfortunate that Colebrooke has been thus misrepresented. It was never the intention of Colebrooke to say that Bhāskara had referred to many previous writers but not to Āryabhaṭa. For the lines written by him, immediately preceding those quoted by Kaye, run as follow: "Towards the close of his treatise of Algebra, Bhāskara informs us, that it is compiled and abridged from the more diffuse works on the same subject, bearing the names of Brahme, (meaning no doubt Brahmagupta), Śrīdhara and Padmanābha; and in the body of his treatise, he has cited a passage of Śrīdhara's Algebra and another of Padmanābha's."† Thus what Colebrooke states is the fact that without the help of the commentators we can know that in his Algebra Bhāskara has referred by names only to three writers preceding him. However, this controversy can arise only if we confine ourselves to Bhāskara's Algebra. But in his bigger work *Siddhānta Śiromaṇi*,‡

* Kaye, Āryabhaṭa, *loc. cit.*, p. 113.

† Henry T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhāscara*, London (1817), p. iv.

‡ In fact, Bhāskara's *Līlāvatī* and *Vījagaṇita* are but earlier parts of this work,

Bhāskara has referred to Āryabhaṭa on several occasions. Two of the instances are of immediate interest to us as being from the *Gaṇita*. In one, * Bhāskara has quoted Āryabhaṭa's value of the ratio of the circumference of a circle to the diameter, *viz.*, 62832/20000. He has himself adopted this value, after a simple reduction, as being an accurate one.† In another instance ‡ Bhāskara refers to the inaccurate expression for the volume of a sphere as found in the *Āryabhaṭīya* (*Gaṇita-pāda*, verse 7).

On a different occassion, referring to Āryabhaṭa's value of π, Kaye remarks that "it was not used by any other Indian Mathematician before the 12th century and that no Indian writer quotes Āryabhaṭa as recording this value."§ This again is wholly untrue. It has been just pointed out that Bhāskara has quoted Āryabhaṭa as recording this value. And against the first part of the above remark, it may be stated that Āryabhaṭa's value of π was applied by Varāhamihira (505 A.D.) and Lalla (c. 578 A.D.). This latter author has declared himself to be a disciple of Āryabhaṭa. It was used by Bhaṭṭotpala (966 A.D.) in his celebrated commentary of Varāhamihira's *Bṛhat Saṃhitā*.|| On the authority of this writer we also learn that it was

* Bhāskara, *Siddhānta-Śiromaṇi*, *Golādhyāya*, *Bhuvanakośa*, verse 52 (*Vāsanā-bhāṣya*).

† *Siddhānta Śiromaṇi*, *Grahagaṇita*, *Candragrahanādhikāra*, verse 3 (*Vāsanā-bhāṣya*) and *Līlāvatī*, ch. iv, verse 40.

‡ Vide *Siddhānta-Śiromaṇi*, *Golādhyāya*, *Bhuvanakośa*, verse 52 (*Vāsanābhāṣya*). "यत् पुन: क्षेत्रफलस्वमूलेन क्षेत्रफलं गुणितं घनफलं स्यादिति, तत् प्रायश्चतुर्वेदाचार्यः परमतमुपन्यस्तवान्" Sudhakara Dvivedi contends : "It is clear by परमत that Bhāskara had not seen the work of Āryabhaṭa ह्यार्य्यभटीय" (*Mahā-siddhānta*, *loc. cit.*, Contents, p. 22 ; also compare *Gaṇaka Taraṅginī*, pp. 36-37). Such an inference seems to be hardly fair. If Bhāskara had not seen the *Āryabhaṭīya* how could he know that in the instance in question Caturvedācārya had recorded the opinion of another. If the latter had recorded it himself, there would have been no need for Bhāskara to add a similar remark. The real fact seems to be this : Caturvedācārya quoted the passage in question without recording his authority. Bhāskara was aware that the result was inaccurate. He added the above remark in order to save Caturvedācārya from the blame of others who may, through their own ignorance, accuse him of establishing such an inaccurate result. However, we know it from many other references by Bhāskara to the *Āryabhaṭīya* that he must have seen this latter work.

§ Kaye, *Indian Math.*, *loc. cit.*, pp. 12-13.

|| *Bṛhat Saṃhitā* with the commentary of Bhaṭṭotpala, ed. Sudhakara Dvivedi, Benares (1895), pp. 53-55.

applied in the recasted *Puliśa-siddhānta*. Thus if reference to Āryabhaṭa's value of π be taken as the determining factor for the date of Āryabhaṭa's *Gaṇita*, wherein alone it is found recorded, it must have to be admitted that the author was anterior to Varāhamihira. There are also other facts to confirm this conclusion.

The quotation from Prithudaka Swami examined

While translating a certain passage of the *Brāhma-sphuṭa-siddhānta*, Colebrooke quoted a comment of Prithudaka Swami wherein the eminent scholiast has pointed out how Āryabhaṭa has denominated certain thing differently from others, "what is termed by us 'diamater less the arrow' is by Āryabhaṭa denominated the greater arrow. For he says, 'In a circle the product of the arrows is equal to the square of the semi-chord of both arcs'." * Kaye thinks this to be "the first purely mathematical quotation" from the *Gaṇita* and then presumes this work to be a product of the 10th century A.D.† So far as is known there lived in this century one Āryabhaṭa, who is reputed as the author of the *Mahā-siddhānta*. But he certainly is not the writer referred to by Prithudaka Swami in connection with the passage in question. For the author of the *Mahā-siddhānta* too has denominated the same thing in the same way as Brahmagupta,‡ *viz.*,

$$\text{Chord} = \sqrt{4\ (\text{diameter}-\text{arrow}) \times \text{arrow}}.$$

In fact, excepting Āryabhaṭa, the author of the *Gaṇita*, all the other Indian mathematicians have used similar denominations in this matter. If Kaye had read Colebrooke's *Algebra* a little more minutely he would have noticed that it contains other passages also from Prithudaka Swami's commentary which have reference to Āryabhaṭa's *Gaṇita*. One (p. 280 fn.) is to the definition of the cube (*Gaṇita*, verse 3) and the other (p. 294 fn.) to the volume of a sphere (*Gaṇita*, verse 7). These also are special to the *Gaṇita* only and thus different from the corresponding matters of the *Mahā-siddhānta*. As has been already pointed out, there are sufficient reasons to think that the *Gaṇita* and

* Colebrooke, *Algebra*, p. 309 footnote. The reference of Prithudaka Swami is to the *Āryabhaṭīya*, *Gaṇita-pāda*, verse 17.

† Kaye, *Āryabhaṭa*, *loc. cit.*, p. 117; Two Āryabhaṭas, *loc. cit.*

‡ *Cf. Mahā-siddhānta*, *loc. cit*, ch xv, verse 98; also *Brāhma-sphuṭa-siddhānta* *loc. cit.*, ch. xii, verse 41.

the *Mahā-siddhānta* cannot be works of the same brain.* So the reference of Prithudaka Swami is to that Āryabhaṭa who was known to and was relentlessly criticised by Brahmagupta and whom he had defended on many occassions against these onslaughts.†

Testimony of Āryabhaṭa's scholiasts

Let us now see what testimony is borne as regards this matter by the scholiasts of the elder Ārybhaṭa. There can be no doubt that their testimony will be very valuable in deciding the date of the *Gaṇita.* Uptil now, we know the names of four scholiasts, Bhāskara (a namesake of the celebrated author of the *Siddhānta-Śiromaṇi*), Sūryadeva, Parameśvara, and Someśvara.‡ All of them have commented upon the *Gaṇita* as a section of the *Āryabhaṭīya.* So we get it on their unanimous testimony that the *Gaṇita* was written in 499 A. D. by the elder Āryabhaṭa. Now Parameśvara lived in the 15th century (*c.* 1430 A.D.).§ It is not impossible for him to get confused about the authorship of a work, however well known, written at least five centuries before. So his testimony may be put aside as not of much weight in settling the present controversy. Such is also the case with Someśvara whose age is not definitely known. The commentary of Bhāskara is not available at present. But he has been "repeatedly cited" by Prithudaka Swami (*c.* 975 A.D.), and by Someśvara.

* *Vide* "Two Āryabhaṭas of Al-Biruni," *loc. cit*, pp. 66-67. There are certain mathematical formulæ which are correct in one book but incorrect in the other. These make it extremely difficult to establish a chronological sequence between the two books, which is essential for the assumption that they were written by the same man.

† Prithudaka Swami seems to be an ardent supporter of the elder Āryabhaṭa. He has not only defended this latter writer against the severe criticism of Brahmagupta, but what is of still greater importance is the fact that, of all the Indian writers he and he alone has supported Āryabhaṭa's theory of the motion of the earth. So it is natural for him to compare and contrast, now and then, the terms and results of Brahmagupta with those of his favourite author.

‡ In the Berlin Library there is a Ms. of a commentary of the *Daśagītikā* by one Bhūtavisnu. As we do not know whether he wrote a commentary of the *Gaṇita* as well, we have ommitted his name from this list.

§ *Vide* Parameśvara's *Goladīpikā*, ed. T. Ganapati Sastri, Trivandrum (1916). His commentary of the *Āryabhaṭīya* has been printed by Kern, *loc. cit.* He has been wrongly called Pramādiśvara by Kern and certain other writers. *cf. Gaṇaka Taraṅginī, loc. cit.*, p. 6 fn.

He thus becomes at least a contemporary of, if not actually anterior to, the younger Āryabhaṭa (c. 950 A.D.).* It is highly improbable that he would display so much ignorance and commit such a mistake in speaking of the work of a contemporary writer, if the *Gaṇita* had happend to be so. For, if the name and fame of a writer is not well spread and if the value of his work is not highly appreciated by the intelligentsia of the time, or at least by an appreciable section of it, there arises no need for writing a commentary on it. On the other hand it is doubtful if the name and the work of the author of the *Mahā-siddhānta* had at all attained such a reputation at that time. For a contemporary writer, Bhaṭṭotpala 966 A.D.), had quoted a host of previous writers including the elder Āryabhaṭa in his commentary of Varāhamihira's *Bṛhat Saṃhitā* but not the younger Āryabhaṭa. This much, however, appears to be without any contradiction that the testimony of Bhāskara can not be overlooked in the same way as can be done with that of Parameśvara. The other known commentator of the *Āryabhaṭīya* is Sūryadeva.† He has quoted Varāhamihira, Lalla and Brahmagupta by name and has himself been quoted by Bhāskarācharya. There are certain other quotations some of which bear a very close resemblance to the corresponding lines of Śrīdhara's *Triśatikā*. They might have been drawn, rather freely, from that work or both might have drawn upon a common source which is at present unknown. If the first inference be true, Sūryadeva must have lived in the period 850-1100 A.D. There are reasons to believe that Sūryadeva was anterior to the younger Āryabhaṭa. It is also noteworthy that on several occasions Sūryadeva has referred to tradition about the teachings of the elder Āryabhaṭa. Sūryadeva has been quoted by Parameśvara.‡

* It should not be overlooked that the time of the younger Āryabhaṭa is somewhat uncertain. Following Dikshit and Sewell, it has been taken here to be *circa* 950 A.D. But, if Dvivedi is right the commentator Bhāskara becomes anterior to the younger Āryabhaṭa and in that case all uncertainty and controversy about the age and authenticity of the *Gaṇita* is settled at once.

† Sūryadeva's commentary has not been printed up to this time. There is an excellent new copy of the Ms. in the Library of the Calcutta University. It was "copied from a Ms. and collated with six others all in palm leaves in Malayālam characters belonging to the office of the Curator for the Publication of Sanskrit Mss., Trivandrum."

‡ Some more valuable references to Āryabhaṭa's *Gaṇita* could be expected from the mathematical works of his disciple Lalla. This work has been referred to by Bhāskarāchārya. Unfortunately it is now lost.

Gaṇita, an integral part of the Āryabhaṭīya: *Internal Evidence*

If we examine more closely the different parts of the *Āryabhaṭīya*, it will be found that the *Gaṇita* forms an integral part of it, and so cannot be separated from it.

(1) It is customary with Sanskrit writers, from times immemorial, to begin writing with an invocation to the Deity worshipped by him or his sect, or to the Deity appropriate to the occasion, and oftentimes to end with a few lines extolling the virtue of mastering the writing. These invocatory verses oftentimes disclose the particular religious faith of the writer. Thus we learn that Āryabhaṭa (the author of the *Āryabhaṭīya*) was a Vedantin; Lalla, Brahmagupta and Śrīdhara were Śaiva; Mahāvīra was a Jaina; Varāhamihira was a Saura. In his earlier days he seems to have been a positivist. It is strange to find that there is no invocatory verse in the *Mahā-siddhānta*. The author begins with an expression of indebtedness to the writers in general that have preceded him. He might have been a positivist of the Sānkhya-Yoga school. His eagerness to conform to the injunctions of orthodox *Smṛitis* will at once preclude of his being doubted for an atheist. The religious and spiritual tenor of the writer disclosed in this way by the invocatory verses of the *Gaṇita* and their want in the *Mahā-siddhānta* will bar all serious attempts to attribute them to a single author.

The first introductory verse of the extant *Āryabhaṭīya*, which is the invocatory verse of the book, runs as follows:—

> "Having prostrated himself before Brahman, one (*in himself but*) many (*in his manifestation*), the true deity, the Supreme divine principle, Āryabhaṭa relates ..." [*Fleet*]*

The opening verse of the *Gaṇita* is :—

> "Having done worship to Brahman [manifested as] the Earth, the Moon, Mercury, Venus, the Sun, Mars, Jupiter, Saturn, and the stars (*more technically the* '*troup of the nakshatras*'), Āryabhaṭa declares..." [*Fleet*]

With these may be read the verse at the end of the *Daśagītikā* and the last but one verse of the whole book. They have also the same tenor as the above two. Those who keep any information of the Vedantic Philosophy, will at once recognise that all of them are from the same brain.

* Fleet, *loc. cit.*

(2) The second line of the first introductory verse runs as :—

"Āryabhaṭa relates three things; Gaṇita (the science of calculation), Kālakriyā (the fixing of time), (and) Gola (the sphere)." [*Fleet*]

Thus as a prelude to his work, there is the promise of Āryabhaṭa to write the *Gaṇita*. So it is very natural to expect that the *Āryabhaṭīya* must contain a *Gaṇita* section. And as has been already stated, the names of the three chapters of the second part of the book are respectively the *Gaṇita*, the *Kāla-Kriyā* and the *Gola*. The relation is too palpable to be overlooked.

(3) It is well known that Āryabhaṭa has used alphabets for a peculiar system of numerical notation. The principle underlying the system has been profounded in the second introductory verse.* There the notational 'places' have been classed as *varga* and *avarga*. The proper significance of these two terms will be clearly understood only on reference to the fourth verse of the *Gaṇita*. Again the arithmetical values of the 'places' and their names have been stated in the second verse of the *Gaṇita*.† So these three verses are closely inter-related.

(4) The tenth verse of the *Daśagītikā* gives the values of $Sin\left(\frac{n\pi}{48}\right)$, for $n=1, 2, 3, \cdots 24$, in succession, without any demonstration. How to divide $\pi/2$ into 24 equal parts as well as the method for calculating the values are indicated in the eleventh and the twelveth verses of the *Gaṇita*. So a knowledge of the latter two verses is indispensable for the proper construction of the table given in the former verse.

(5) In the *Daśagītikā* (verse 4) is indicated a method of calculating the planetery orbits and in the *Kāla-kriyā* (verses 19, 25) and the *Gola* (verses 39, 40), there are applications of the diameters of these orbits. Again the verse 5 of the *Daśagītikā* states the length of the diameter of the Earth and the verse 9 of the same section states the circumference of the *spiritus vector* (earth's atmosphere). To be self-contained, the *Āryabhaṭīya* must contain a method of calculating the diameter of a sphere from its circumference or *vice versa*. The value of π necessary for the purpose is contained in the *Gaṇita* section of

* Fleet, *loc. cit.*

† The classification of notational places into *varga* and *avarga* might have been also due to simple phonetic resemblance to a similar and older classification of the Sanskrit consonants. Even in that case the other arguments will stand.

the book. And we learn it from Parameśvara that Āryabhaṭa did actually use the same value for a similar purpose.*

The above instances will be considered sufficient, we hope, to prove that the different parts of the *Āryabhaṭīya* are inter-dependent to almost an indispensable extent. So the *Gaṇita* section of the book cannot be separated from its other sections and attributed to a later namesake as had been attempted by Kaye.

External Evidence

Under this sub-head, we shall collect evidence from the treatises other than the *Āryabhaṭīya,*—omitting, of course, the works of the scholiasts which have been already noticed—which will go to prove, directly or indirectly, that the *Gaṇita* must be the work of that Aryabhaṭa. who was anterior to Brahmagupta. He is the author of the *Āryabhaṭīya.*

(1) The most relentless critic known of the author of the *Āryabhaṭīya* was Brahmagupta. Having criticised him mercilessly to some length in the *Brāhma-sphuṭa-siddhānta*, Brahmagupta has remarked †:—

> "As Āryabhaṭa does not know either of the *Gaṇita*, the *Kāla* and the *Gola*, so the faults of these are not pointed out separately by me."

Thus we find that Brahmagupta has stated the *Gaṇita* to be a section of the work of Āryabhaṭa. That he had in view the author of the *Āryabhaṭīya* has not been and, indeed, cannot be disputed.

(2) It has been already pointed out that the second part of the the *Āryabhaṭīya* has been called *Āryāṣṭaśata* in the *Brāhma-sphuṭa-siddhānta*, because it contains 108 verses in the Āryā metre. All the manuscripts of the *Āryabhaṭīya* that have been recovered up to this

* *Vide Goladīpikā, loc. cit,* verse 30.

† *Brāhma-sphuṭa-siddhānta,* ch. xi, verse 43.

जानात्येकमपि यतो नार्यभटो गणितकालगोलानाम् ।
न मया प्रोक्तानि ततः पृथक् पृथक् दूषणान्येषाम् ॥

It is noteworthy that the titles and their sequence are almost the same as are in the *Āryabhaṭīya.* The slight difference in the title of the second chapter, —omission of *Kriyā,*—is immaterial. Most probably Brahmagupta had to cut short the full title for the sake of easy construction of the metre.

time, have been found to contain 108 verses, for this part. This is remarkably significant. If the *Gaṇita* is to be excluded, as has been suggested by Kaye, on the basis of an obscure statement of Al-Biruni, we shall have to admit either that the name *Āryāṣṭaśata* is meaningless, or that all the available Mss. are incomplete and misleading. In any case such a bold assertion will lead to creation of fresh difficulties in the hope of meeting one about the reality of which there hangs much doubt.

(3) While referring to Āryabhaṭa's value of π (=62832/20000), Bhāskara has remarked that Brahmagupta, Śrīdharācārya and others had taken the value of π to be approximately equal to the square root of ten 'for simplicity's sake' (*sukhārtham*) not because they were ignorant of the other accurate value.* We do not know what was the source of information of Bhāskara in making this assertion unless it be a general inference from the fact that each of those writers was acquainted with the works of Āryabhaṭa (*vide infra*). But this remark has a very important bearing on the subject under discussion. For it testifies that, in the opinion of Bhāskara, the *Gaṇita* which alone contains that value of π, is a work of Āryabhaṭa who preceded Brahmagupta and others.

(4) It has been stated by Al-Biruni that in the "amended edition" of his *Khaṇḍakhādyaka*, Brahmagupta has applied Āryabhaṭa's value of π.† In his other works, he has taken $\pi = \sqrt{10}$.

* *Siddhānta-Śiromaṇi, Golādhyāya, Bhuvanakośa*, verse 52 (*Vāsanābhāṣya*).

"यतोऽयुतद्वयव्यासे २०००० त्रिकाग्न्यष्टवसुमितः ६२८३२ परिधिरार्यभटाद्यैरुक्तः। यत् पुनः श्रीधराचार्यब्रह्मगुप्तादिभिर्व्यासवर्गाद्दशगुणात् पदं परिधिः स्थूलोऽप्युक्तः, स सुखार्थम्। नहि ते न जानन्तीति।"

Who are the others besides Brahmagupta and Śrīdharācārya who adopted Āryabhaṭa's value of π? It may be well to point out that Bhāskara himself has used similar rough approximations on several occasions though he was aware of the more accurate values. For instance see *Siddhānta Śiromaṇi*, *Spaṣṭādhikāra*, verses 17 and 73 (*Vāsanābhāṣya*).

† Al-Biruni's *India*, vol. I, 312; compare also II.67. This statement of Al-Biruni seems to be quite authenticated. For, he was fortunate to secure, while in India, Mss. of Brahmagupta's *Khaṇḍakhādyaka* together with "amended edition of it" by the author and its commentary by Balabhadra (II, 187). Further, Al-Biruni later on published a "new and amended edition" of the first Arabic translation of this book. *Cf.* I. 156; II. 46, 49.

Is this amended edition of *Khaṇḍakhādyaka* also the source of information to Bhāskara about Brahmagupta's knowledge of Āryabhaṭa's value of π, as noted above?

(5) Āryabhaṭa's works were introduced in Arabia by the Hindu scholars who accompanied the embassy from Sindh to the Khalifa Almansur about 770 A.D. Alfazari, Yakub ibn Tarik and Abu Alhasan of Ahwās learnt of Hindu Astronomy from them. These scholars must have carried with them the Mss. of the *Gaṇita* or at least a knowledge of its subject matter; and must have taught it to their Arab disciples. For we learn it on the authority of Al-Biruni that Yakub ibn Tarik has recorded in his book, *Compositæ Sphærarum* (or the *Composition of the Spheres*) "*on the authority of his Hindu informant,*" certain datas which contain an application of the value of π found in the *Gaṇita*.* It is still more remarkable that in this book Yakub has recorded Āryabhaṭa's values about the dimensions of the Earth.†

(6) With reference to Āryabhaṭa's value of π, Mohammad Ibn Mūsā al-Khowārizmī (825 A.D.) has remarked that it is the *value of the Indian astronomers*.‡

After these facts there cannot be any doubt that the *Gaṇita* existed at least before 750 A.D. And as we know it definitely that there lived in India one Āryabhaṭa (*b.* 476 A.D.) before that age, the *Gaṇita* must be his work. Hence Kaye's doubt about its authenticity and his conjecture about its being a work of the 10th century A.D., are absolutely false. It should be stated in fairness to Kaye that he seems to have somewhat modified his early opinion. For in his *Indian Mathematics*, published in 1915, in addition to a record of this doubt (p. 11 footnote), he has also put a sub-heading "*Āryabhaṭa's Gaṇita*—(circa A.D. 520)" (p. 47). He knows it best how he calculated the date 520 A.D.

* Al-Biruni's *India*, I. 169. This reference has also been used by Kaye but to a different end. He has suppressed the passage put within inverted commas and italicised; *cf.* Kaye, Āryabhaṭa, *loc. cit.*, p. 122.

† *Ibid*, II. 67; "He (Yakub) had drawn his information from the well known Hindu scholar who, A.H. 161 (? 154), accompanied an embassy to Bagdad."

‡ *Vide* Rodet, *loc. cit.*, p. 414; "Al-Khowārizmī cite cette valuer 62832/20000 comme due aux "astronomes" indiens;" compare also, "L Algebre d' Al-khwarizmi et les methodes Indienne et Grecque," *Journ. Asiat.* (1878). Here again Kaye is guilty of suppression,—probably also of unfair marshalling of facts. He says "M. Ibn Musa gives the value 62832/20000 but also gives a summary of Archimedes' proof and it is absolutely certain that M. Ibn Musa did not copy this from the Hindus." (Kaye, Āryabhaṭa, *loc. cit.*, p. 122; *cf.* also *Indian Mathematics*, pp. 41-42).

3

ON THE STRESS AND STRAIN IN AN ELASTIC PLATE HAVING TWO INFINITELY LONG RECTANGULAR EDGES UNDER THE ACTION OF A COUPLE AT THE CORNER*

BY

N. M. BASU AND H. M. SEN-GUPTA

The problem of the determination of the stress and strain in an elastic plate under forces applied in its plane has been discussed at great length by Michell † in a number of memoirs. One of the solutions due to him is that of the problem of a semi-infinite elastic plate subjected to pressure at a point of the straight edge. He has also given a solution for an elastic plate bounded by two infinitely long straight edges inclined at any angle and acted on by a force at the angle. But both of these solutions are defective in as much as the values obtained for the displacements become infinitely large at infinite distances. The effect of a uniform tangential traction applied over one half of the infinite straight edge of a semi-infinite elastic plate has also been investigated but the solution obtained is likewise open to objection since not only the displacements but the stress as well tend to become infinitely large at infinite distances.‡

In the present paper the effect of a couple applied at the corner of a thin elastic plate having two mutually perpendicular infinitely long straight edges has been investigated. The importance of the result, which is believed to be new, lies in the fact that the stress-components as well as the displacements vanish at infinity and the solution is therefore free from the objections mentioned above.

2. It is well-known that the problem of the equilibrium of an elastic plate under no body forces is identical with the problem of plane strain under certain conditions which may be taken as valid

* An abstract of this paper was read at the Indian Science Congress, 1927.

† See *Proceedings of the London Mathematical Society*, Vols. 31, 32 and 33.

‡ *Cf.* L. N. G. Filon, *Phil. Trans. Roy. Soc.* (Ser. A), Vol. 198.

when the plate is very thin and is free from traction over the faces. The values of the displacements and stress-components that are obtained by treating the problem as one of plane strain are their average values, the average being taken over the thickness of the plate, and will therefore be more and more nearly in agreement with the actual values as the thickness of the plate is more and more diminished. The state of stress and strain is then determined by a single function called the stress function.

If χ denote the stress-function, then the following results are well-known.*

(1) $$\nabla_1^4 \chi = 0$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(2) The stress-components in cartesian co-ordinates are given by

$$X_x = \frac{\partial^2 \chi}{\partial y^2}, \quad Y_y = \frac{\partial^2 \chi}{\partial x^2}, \quad X_y = -\frac{\partial^2 \chi}{\partial x \partial y}$$

(3) The stress-components in polar co-ordinates are given by

$$\widehat{rr} = \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \chi}{\partial r}$$

$$\widehat{\theta\theta} = \frac{\partial^2 \chi}{\partial r^2}$$

$$\widehat{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial \chi}{\partial \theta}\right)$$

(4) The cubical dilatation is given by

$$\Delta = \frac{\nabla_1^2 \chi}{2(\lambda' + \mu)},$$

where $$\lambda' = \frac{2\lambda\mu}{\lambda + 2\mu}$$

* See Love's *Theory of Elasticity*—Chap. 9.

and λ and μ are the elastic constants of the material of the plate.

(5) The rotation ω is the harmonic conjugate to

$$\frac{\lambda'+2\mu}{2\mu}\,\Delta.$$

(6) The cartesian components of displacement are given by

$$u=\frac{1}{2\mu}\left\{-\frac{\partial\chi}{\partial x}+\xi\right\},$$

$$v=\frac{1}{2\mu}\left\{-\frac{\partial\chi}{\partial y}+\eta\right\},$$

where ξ and η are determined by the relation

$$\xi+i\eta=\int\left\{(\lambda'+2\mu)\,\Delta+i2\mu\omega\right\}d(x+iy).$$

3. Take cartesian axes along the straight edges and let the x—axis be the initial line for polar co-ordinates.

Consider the stress-function

$$\chi=-\frac{Ly^2}{r^2}=-L\sin^2\theta,$$

where L is a constant.

Then χ obviously satisfies the equation $\nabla_1^4\chi=0$. Using the results given in Art. 2, the following values are easily obtained :—

$$\widehat{rr}=-\frac{2L}{r^2}\cos 2\theta$$

$$\widehat{\theta\theta}=0$$

$$\widehat{r\theta}=-\frac{L}{r^2}\sin 2\theta$$

$$X_x=\frac{2Lx^2}{r^6}(3y^2-x^2)$$

$$Y_y=\frac{2Ly^2}{r^6}(y^2-3x^2)$$

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$$\widehat{xy} = \frac{4Lxy}{r^6}(y^2-x^2)$$

$$\Delta = \frac{L}{\lambda'+\mu} \cdot \frac{y^2-x^2}{r^6}$$

$$\widehat{\omega} = \frac{(\lambda'+2\mu)L}{\mu(\lambda'+\mu)} \cdot \frac{xy}{r^4}$$

$$\xi = \frac{\lambda'+2\mu}{\lambda'+\mu} L\frac{x}{r^2}$$

$$\eta = -\frac{\lambda'+2\mu}{\lambda'+\mu} L\frac{y}{r^2},$$

$$u = \frac{Lx}{2\mu(\lambda'+\mu)r^4}\left\{(\lambda'+2\mu)x^2-\lambda'y^2\right\}$$

$$v = \frac{Ly}{2\mu(\lambda'+\mu)r^4}\left\{\lambda'x^2-(\lambda'+2\mu)y^2\right\}$$

$$u_r = \frac{ux+vy}{r} = \frac{L(\lambda'+2\mu)}{2\mu(\lambda'+\mu)} \cdot \frac{\cos 2\theta}{r}$$

$$u_\theta = \frac{vx-uy}{r} = -\frac{L\sin 2\theta}{2(\lambda'+\mu)r},$$

where u_r and u_θ are the radial and transverse displacements respectively.

4. With the origin as centre and any radius r draw a circle and consider the resultant stress exerted by the quadrant bounded by this circle and the edges on the rest of the plate.

If X, Y denote the components of the resultant stress parallel to the edges and if N denote the moment of the stress about the origin 0, we have,

$$X = -\int_0^{\frac{\pi}{2}}\left(\widehat{rr}\cos\theta - \widehat{r\theta}\sin\theta\right)rd\theta$$

$$= \frac{L}{r}\int_0^{\frac{\pi}{2}}\left(2\cos\theta\cos 2\theta - \sin\theta\sin 2\theta\right)d\theta$$

$$= 0$$

$$\mathrm{Y} = -\int_0^{\frac{\pi}{2}} \left(\widehat{rr}\sin\theta + \widehat{r\theta}\cos\theta\right) r d\theta$$

$$= \frac{\mathrm{L}}{r}\int_0^{\frac{\pi}{2}} \left(2\sin\theta\cos 2\theta + \cos\theta\sin 2\theta\right) d\theta$$

$$= 0$$

$$\mathrm{N} = -\int_0^{\frac{\pi}{2}} \widehat{r\theta}\, r^2 d\theta$$

$$= \mathrm{L}\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta$$

$$= \mathrm{L}$$

Thus the resultant stress is a couple of moment L, whatever the radius r may be. We may therefore suppose that a portion of the plate cut off by a circle of infinitely small radius is removed and a couple of moment L is applied at the corner of the plate. In order that the results of Art. 3 may correctly represent the state of stress and strain in the plate under the action of this couple, the straight edges must be free from traction and the stress-components and displacements must vanish at infinity. That the edges are free from traction is easily verified since $\widehat{\theta\theta}=0$ everywhere and $\widehat{r\theta}=0$ over $\theta=0$ and $\theta=\frac{\pi}{2}$. It is also seen that the stress varies inversely as the square of the distance while the displacement varies inversely as the distance from the origin and they therefore vanish at infinity. Thus the results of Art. 3 represent the solution of the problem discussed in this paper.

5. It may be noted that for points on the edges, there is no rotation. Such points are displaced along the edges and, when L is positive, points on the x-axis are displaced away from the origin while points on the y-axis are displaced towards the origin, the magnitudes of the displacements at equal distances being equal. The radii drawn from the origin are subjected to shearing stress only. Elements along the middle radius $\left(\theta=\frac{\pi}{4}\right)$ undergo no rotation and receive only transverse

displacement towards the side of the x-axis. All points on the side of this line towards the x-axis have their distances from the origin increased while all points on the opposite side have their distances diminished.

6. The equations of the lines of stress are easily obtained. If ds be an element of a line of stress at the point (x, y), so that the direction cosines of the normal to this element are

$$\frac{dy}{ds} \text{ and } -\frac{dx}{ds},$$

the differential equation of the line of stress is:

$$\frac{X_x \frac{dy}{ds} - X_y \frac{dx}{ds}}{\frac{dy}{ds}} = \frac{X_y \frac{dy}{ds} - Y_y \frac{dx}{ds}}{-\frac{dx}{ds}}$$

or

$$X_x - X_y \Big/ \frac{dy}{dx} = Y_y - X_y \frac{dy}{dx}$$

or

$$\left(\frac{dy}{dx}\right)^2 + \frac{X_x - Y_y}{X_y} \frac{dy}{dx} - 1 = 0$$

Substituting the values of X_x, Y_y and X_y, we have

$$\left(\frac{dy}{dx}\right)^2 - \frac{6x^2y^2 - x^4 - y^4}{2xy(x^2 - y^2)} \frac{dy}{dx} - 1 = 0$$

or

$$\left(\frac{dy}{dx} + \frac{x^2 - y^2}{2xy}\right)\left(\frac{dy}{dx} - \frac{2xy}{x^2 - y^2}\right) = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x^2 - y^2}{2xy} \text{ or } \frac{2xy}{x^2 - y^2}$$

Solving the above equations, we get

$$\frac{x^2 + y^2}{2x} = A,$$

and

$$\frac{x^2 + y^2}{2y} = B,$$

where A and B are arbitrary constants. Thus the lines of stress are the two systems of orthogonal circles having their centres on the axes of x and y and touching respectively the axes of y and x at the origin.

4

TIME BY ALTITUDE IN INDAIN ASTRONOMY

BY

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1. Determination of time by altitude is ascribed to the Arabs by European scholars. In the article on "Measurement of Time" in the *Encyclopaedia Britannica* it is stated that "in the year 829 we find it stated that at the commencement of the solar eclipse on the 30th November the altitude of the sun was 7° and at the end 24° as observed at Bagdad by Ahmad ibn Abdullah, called Habash. This seems to be the earliest determination of time by altitudes." Again in the article on "Astronomy" in the Section 'History of Astronomy' it is stated that ,'George Purbach (1423-1461) introduced into Europe the method of determining time by altitudes employed by Ibn Junius (950-1008).' This short paper seeks to establish the priority of Indian astronomers and the indebtedness of the Arabs to their Indian teachers in this respect.

2. The problem was first attacked by Aryabhata but with a partial success, and was solved by Varahamihira although he did not discard the wrong solution by Aryabhata. The solution in the proper form was first given by Brahmagupta. The dates of these authors are :—

A. D. 449 of Aryabhata.
,, 550 of Varahamihira.
,, 628 of Brahmagupta.

3. The original passages from these authors may be thus translated :—

(*a*) "The sine in the diurnal circle, of the time elapsed since sunrise (or of the time to elapse till sunset), multiplied by the sine of the

co-latitude and divided by the radius is the sine of the altitude of the sun." *Gola*, 28.

Symbolically it means,

$$\frac{R \sin t \times R \cos \delta}{R} \times \frac{R \cos \phi}{R} = R \cos z,$$

where t is the time elapsed from sunrise, ϕ the latitude of the station, δ the sun's declination and z his zenith distance. This is only an approximate rule.

(*b*) "Take the square root of the sum of the squares of the shadow and of twelve; multiply it by the sine of the co-latitude and by the product divide 172800; the quotient is called the first sine.

"Multiply the sine of the latitude by the sine of the sun's declination of that day and divide by the sine of co-latitude; put the result down separately. Deduct it from the first sine if the sun has northern declination; add it, if he has southern declination.

"Multiply the first sine so modified and the sine which has been put down separately by 240 and divide by the double day radius; the two corresponding arcs have to be added to each other if the sun has northern declination, while they have to be deducted in the case of southern declination. The resulting degrees divided by six give the nadikas."

Pancha Siddhantika - IV, 45-47.

Symbolically these rules are as follows:—

(1) Shadow $= 12 \tan z$, and $\sqrt{\text{shadow}^2 + 12^2} = 12 \sec z$.

$$\text{The first sine} = \frac{172800}{12 \sec z \times R \cos \phi} = \frac{12 \times R \times R}{12 \sec z \times R \cos \phi},$$

(Since Varahamihira's radius $= 120$)

(2) $\dfrac{R \sin \phi \times R \sin \delta}{R \cos \phi}$ is the second sine,

then let

$$\left(\frac{12\times R\times R}{12\sec z\times R\cos\phi}-\frac{R\sin\phi\times R\sin\delta}{R\cos\phi}\right)\times\frac{2R}{2R\cos\delta}=R\sin h'$$

and

$$\frac{R\sin\phi\times R\sin\delta}{R\cos\phi}\times\frac{2R}{2R\cos\delta}=R\sin h'';$$

here $h'+h''$ gives the time elapsed from sun rise.

(c) Brahmagupta's stanzas need not be quoted in full. He directly speaks of the hour angle from the sun's altitude. His rule may be thus expressed :—

$$R-\left(\frac{12\sec\phi\times R}{12\sec z}-\frac{R\sin\delta\times 12\tan\phi}{12}\right)\times\frac{R}{R\cos\delta}=R\text{ vers }h,$$

where h is the hour angle of the sun.

B. S., III, 38-40.

These rules are readily seen to be equivalent to the well known equation

$$\cos z=\sin\phi\sin\delta+\cos\phi\cos\delta\cos h.$$

The priority of Indian Astronomers in this respect is thus established.

4. As to the indebtedness of the Arabs to the Indian Astronomers we make the following quotations from Dr. E. C. Sachau's annotations to his translation of Alberuni's *India* :—

(1) "Brahmagupta holds a remarkable place in the history of Eastern civilisation. It was he who taught the Arabs astronomy before they became acquainted with Ptolemy; for the famous *Sindhind* of Arabian literature, frequently mentioned but not yet brought to light, is a translation of his *Brahmasiddhanta* ; and the only other book on Indian Astronomy, called *Alarkand*, which they knew, was a translation of his *Khandakhadyaka*."

(2) "Alfazari is one of the fathers of Arabian literature, the first propagator of Indian astronomy among the Arabs......... his book contained the cycles of the planets as derived from Hindu Scholars, the members of an embassy from some part of Sindh, who called on the Khalif Almansur in A.D. 771."

(3) "In the first fifty years of Ābbaside rule there were two periods in which the Arabs learned from India, the first under Mansur (A.D. 753-774) chiefly Astronomy, and secondly under Harun (786-808), by the special influence of the ministerial family Barmak, who till 803 ruled the Muslim world, specially medicine and astrology."

Lastly we may mention that during the rule of Khalif Mamun (813-847), all standard works on Science, Philosophy, medicine, etc., in all the western and eastern languages were translated into Arabic. Amongst the translators was one Duban the Brahmin who translated from Sanskrit. (*Vide* Ameerali's *History of the Saracens*). It is noteworthy that the event referred to in the article of the *Encyclopaedia Britannica* on "Measurement of Time," happened in (829 A. D.) the reign of this Khalif.

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ON MUTUALLY SELF-POLAR TRIANGLES

BY

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1. Starting* with the idea of pole and polar in regard to a triangle, we explain the concept indicated in the title, deduce therefrom the configuration of Pappus, show how the configuration of critic centres may be generated independently of the theory of the cubic, and we add an independent verification that the mutual self-polarity of two triangles implies and is implied by the commutative law of multiplication.

2. Given a triangle ABC and any point P in its plane, if P_1, P_2, P_3 be the projections of P from the vertices on the opposite sides then it is known that the fourth harmonics of P_1 with respect to B, C of P_2 with respect to C, A and of P_3 with respect to A, B lie on a line called the polar line of P in regard to the triangle ABC.† This relation between the points and the lines of the plane of ABC is a correlation of which the united elements are the elements of the triangle itself. Furthur, we may consider the point and the line as pole and polar in regard to the degenerate cubic consisting of the sides of the triangle, and equally in regard to the degenerate class-cubic consisting of the vertices of the triangle.

When we seek on the line for any points whose polars pass through P, we find there are just two such points Q, R of which not only is it true that the polars of both pass through P, but equally does the polar of either pass through the other. We may thus speak of the triangle PQR as being self polar in regard to the triangle ABC, in the

* This paper was read before the Indian Science Congress on January 6, 1927.

† *Cf*: Salmon-chemin pp. 204-5.

sense that each of its vertices is the pole of the opposite side: and given ABC such a triangle as PQR is uniquely fixed as soon as one of its elements is assigned.

It is a fact, easily verified with the help of sections and projections, that when the triangle PQR is self polar to the triangle ABC, equally is it true that ABC is also self polar in regard to PQR. Thus the relation of self polarity as between two triangles is a mutual one, and we are led to speak of two mutually self-polar triangles. Each vertex and the opposite side of either triangle are pole and polar in regard to the other triangle considered as a degenerate point-cubic, and also when considered as a degenerate class-cubic.

3. It is known that when a triangle ABC is in perspective with a triangle A′B′C′, and also with the triangle B′C′A′, so is it equally in perspective with C′A′B′, and the two triangles ABC and A′B′C′ are said to be in three-fold cyclic perspective.* The three centres of perspective then form a third triangle such that the relations between the three triangles are symmetrical; in other words, any two of them are in cyclic perspective with the vertices of the third for centres. The figure consisting of the nine vertices of three such triangles is the figure that establishes the truth of the Theorem of Pappus.†

4. The following results may now be established.

(*a*) If a triangle T_1 is self-polar with respect to a triangle T_2 and also self-polar with respect to a conic C, so is it also self-polar in regard to T'_2, the reciprocal of T_2 with respect to C.

(*b*) A related result is that if a triangle T_1 is self-polar with respect to each of two triangles T_2, T_2' which are situated in one way perspective position, so is it self-polar with respect to the conic for which T_2 reciprocates into T_2'.

(*c*) If a triangle T_0 is self-polar with respect to each of two triangles T_1 and T_2, these last are necessarily in threefold cyclic perspective, and the centres of perspective also form a triangle T_3 which is as well self-polar to T_0 and the vertices of T_1, T_2, T_3 complete a Pappus configuration.

* *Cf.* Rosanes, *Math. Ann.* Vol. 2, page 549.

† *Cf* Baker, *Principles*, Vol. 1, page 47

(*d*) Conversely when two triangles are in cyclic perspective there exists a single further triangle which is at once self polar to both and is also self polar for each of the three conics with respect to which the one triangle reciprocates into the other; these three conics thus have a single common self conjugate triangle.

(*e*) In continuation of the result (*c*) above, if T_3 is also self polar to T_1 and T_2 then T_1, T_2 are in sixfold perspective, and the four triangles T_0, T_1 T_2, T_3 are symmetrically related; that is each one of them is self-polar to each of the other three, and every two of them are in sixfold perspective with the vertices of the other two for centres.

When such a position arises, the vertices of the four triangles are the critic centres of a cubic curve.

(*f*) In connection with the result (*c*) above we have the following result. Given two sets of three lines meeting in nine points the necessary and sufficient condition for a third set of three lines to pass through the nine points is the existence of a triangle self polar in regard to the two triangles formed by the two given sets of three lines.

5. It is known that the existence of the configuration of Pappus is the Geometric equivalent of the Algebraic Law of commutative multiplication. We have seen above that this configuration is capable of being deduced from the idea of one triangle being self polar to another. It follows therefore that the mutual self polarity of two triangles implies and is implied by the law of multiplication. That this is indeed so is capable of easy verification as we proceed to show.

Let * ABC be any triangle and P any point in its plane, such that the linear relation amongst the four coplanar points is $A+B+C+P=O$. The projection of P from A on B C is then $B+C$, and its harmonic with respect to B and C is $B-C$. The three points thus obtained $B-C$, $C-A$ and $A-B$ then lie on a single line namely the polar of P in regard to the triangle ABC. The general point on this line may be taken as $\Sigma=\theta(C-A)+\phi(A-B)$ so that the projection of Σ from C on AB is $(\phi-\theta)$ $A-\phi$ B and its harmonic in regard to A, B is $(\phi-\theta)A+\phi B$; so also the projection of Σ from B on CA is $(\phi-\theta)A+\theta C$ and its harmonic in regard to A, C is $(\phi-\theta)A-\theta C$.

* The symbolism will be familiar to readers of Baker, *Principles*, vol. 1 chapter 1, section 3.

The join of these two harmonics is the polar of Σ and will pass through P which is $A+B+C$ if we can show that a linear function of $(\phi-\theta)A-\theta C$ and $A+B+C$ is identical with $(\phi-\theta)A+\phi B$. For this we require that

$$[(\phi-\theta)A-\theta\, C]+\theta[A+B+C]$$

which is same as $\phi A+\theta B$ should be identical with $(\phi-\theta)A+\phi B$. Equating ϕ times the former symbol to θ times the latter, we obtain

$$\phi\theta=\theta\phi, \quad \phi^2=\theta(\phi-\theta).$$

Thus when the former equality holds the latter relation yields the two possible positions of Σ, which completes the verification.

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SIMPLEXES IN n-DIMENSIONS

BY

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In the present paper an attempt has been made to study the figure by methods, simple and general, and certain properties which are analogous to those of tetrahedrons have been deduced. Indeed, many theorems which are seen only partially in the ordinary geometries appear in their true nature in the geometries of higher dimensions. The following treatment is however restricted to Euclidean spaces only.

My best thanks are due to Dr. S. M. Ganguli who kindly suggested the investigation to me.

1. A simplex of the nth order is defined as a figure consisting of $n+1$ independent points in an n-space.* It may also be considered as determined by $n+1$ linear loci (spaces of $n-1$ dimensions)

$$L^{(j)}, \quad (j=1, 2, \ldots\ldots n+1),$$

n of which meet in a point, a vertex. Through each vertex pass n edges, $\frac{1}{2}n(n-1)$ planes, ... the intersections of every $n-1$, $n-2$, ... linear loci passing through the vertex. The linear loci may be called the faces of the simplex. The numbers of vertices, edges, planes, etc. of the simplex are connected by the relation

$$1-(n+1)_1+(n+1)_2-(n+1)_3+\ldots\ldots\pm(n+1)_{n+1}=0\,\dagger,$$

* Vide *Proceedings of the London Math. Soc.*, Vols. XVIII, XIX. The figure is named a simplicissimum. Also '*Theorie der vielfachen Kontinuitat*' by L. Schläfli, where we have a polyschem.

† An interesting discussion of this law is to be found in Schoute's *Mehrdimensionale Geomatrie*, Vol. II. § 2; also in Schläfli § 10.

where $(n+1)_r$ denotes the number of combinations of the vertices taken r at a time, this being the expansion of $(1-1)^{n+1}$, where n is any positive integer.

$$\text{Let} \quad L^{(j)} \equiv l_{j1}x_1 + l_{j2}x_2 + l_{j3}x_3 + \ldots\ldots + l_{jn}x_n + l_{j(n+1)} = 0,$$

where $x_1, x_2, \ldots x_n$ are the co-ordinates of a point referred to a system of n concurrent lines mutually orthogonal.

(It may be seen that the co-ordinate axes divide the n-space about the origin in 2^n parts, which are associated in 2^{n-1} pairs, the two parts of a pair being opposite to one another.)

Let us suppose, for simplicity's sake, that

$$\Sigma l^2_{1i} = \Sigma l^2_{2i} = \ldots\ldots = \Sigma l^2_{(n+1)i} = 1, \quad (i = 1, 2, 3 \ldots\ldots n)$$

and assuming the conventions made in the ordinary geometry we write the equations to the loci so that

$$l_{j(n+1)}, \quad (j = 1, 2, \ldots\ldots n+1)$$

are all positive. Consider the determinant

$$\begin{vmatrix} l_{11} & l_{12} & \ldots & \ldots & l_{1(n+1)} \\ l_{21} & l_{22} & \ldots & \ldots & l_{2(n+1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ l_{(n+1)1} & l_{(n+1)2} & \ldots & & l_{(n+1)(n+1)} \end{vmatrix} \equiv \Delta \text{(say)}$$

If Δ is zero, all the faces pass through a common point ; so we suppose Δ to be always different from zero.

Let us denote the vertex

$$(L^{(1)} L^{(2)} \ldots\ldots L^{(j-1)} L^{(j+1)} \ldots\ldots L^{(n+1)})$$

by A_j, with similar notations for others. Then the co-ordinates of A_j are given by

$$\frac{x_1}{L_{j1}}=\frac{x_2}{L_{j2}}=\ldots\ldots=\frac{x_n}{L_{jn}}=\frac{1}{L_{j(n+1)}},$$

where L_{jr} is the first minor of l_{jr} in the determinant Δ (with the proper sign), *i.e.*,

$$L_{jr}=\frac{\partial\Delta}{\partial l_{jr}}.$$

If V be the content of the simplex then we have, as in the ordinary geometry,

$$n!V=\frac{\Delta^n}{\Pi L_{j(n+1)}},$$

where Π denotes the continued product

and $j=1, 2, \ldots\ldots n+1$.

Now the content of any simplex of the $(n-1)$th order whose vertices are all the vertices but one of the simplex of the nth order is equal to the sum of the projections on it of the other n simplexes of the same $(n-1)$th order formed in a similar manner. Then denoting by V_j the content of the simplex of the $(n-1)$th order whose vertices are all the vertices but A_j we have *

$$V_1\cos L^{(j)}L^{(1)}+V_2\cos L^{(j)}L^{(2)}+\ldots\ldots$$

$$+V_{j-1}\cos L^{(j)}L^{(j-1)}-V_j+V_{j+1}\cos L^{(j)}L^{(j+1)}+\ldots\ldots$$

$$+V_{n+1}\cos L^{(j)}L^{(n+1)}=0, \quad \text{(I)}$$

where $L^{(j)}L^{(r)}$ denotes the angle between the faces $L^{(j)}$ and $L^{(r)}$.

* Salmon's geometry of three dimensions, § 56.

There are $n+1$ such relations corresponding to

$$j=1,\ 2,\ldots\ldots n+1$$

and any n of them will determine the ratios of

$$V_1,\ V_2,\ldots\ldots V_{n+1}$$

to one another in terms of the angles between the faces. Again since the angle between any two faces is supplement of the angle between their normals drawn from the origin, the above relations can be written as

$$V_1\left(\sum_{i=1}^{i=n} l_{ji}l_{1i}\right)+V_2\left(\sum_{i=1}^{i=n} l_{ji}l_{2i}\right)+\cdots$$

$$+V_{j-1}\left(\sum_{i=1}^{i=n} l_{ji}l_{(j-1)i}\right)+V_j$$

$$+V_{j+1}\left(\sum_{i=1}^{i=n} l_{ji}l_{(j+1)i}\right)+\ldots\ldots$$

$$+V_{n+1}\left(\sum_{i=1}^{i=n} l_{ji}l_{(n+1)i}\right)=0.$$

Eliminating these V's from these $n+1$ relations corresponding to $j=1,\ 2,\ldots\ldots n+1$, we have

$$\begin{vmatrix} 1 & \Sigma l_{1i}l_{2i} & \Sigma l_{1i}l_{3i} & \ldots & \Sigma l_{1i}l_{(n+1)i} \\ \Sigma l_{2i}l_{1i} & 1 & \Sigma l_{2i}l_{3i} & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \Sigma l_{(n+1)i}l_{1i} & \ldots & \ldots & \ldots & 1 \end{vmatrix}=0.$$

But since the determinant vanishes the first minors of any row are respectively proportional to the corresponding first minors of any other.

Hence the minors of this determinant are proportional to the contents of the simplexes of the $(n-1)$th order. But

$$\begin{vmatrix} 1 & \Sigma l_{1i}l_{2i} & \dots & \Sigma l_{1i}l_{ni} \\ \Sigma l_{2i}l_{1i} & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \Sigma l_{ni}l_{1i} & \dots & \dots & 1 \end{vmatrix} = L^2_{(n+1)(n+1)}.$$

Therefore $$V_1, V_2, \dots\dots V_{n+1}$$

are respectively proportional to

$$L^2_{1(n+1)}, \quad L^2_{2(n+1)}, \dots\dots L^2_{(n+1)(n+1)}.$$

Let a point C be taken within the simplex whose co-ordinates are

$$\bar{x}_i, \ (i=1, 2, \dots\dots n).$$

Lines drawn from C perpendicular to the faces will be of the form

$$\frac{x_1-\bar{x}_1}{l_{j1}}=\frac{x_2-\bar{x}_2}{l_{j2}}=\dots\dots\frac{x_n-\bar{x}_n}{l_{jn}}=\rho_j \text{ (say)}.$$

There are $n+1$ such lines corresponding to $j=1, 2, \dots\dots n+1$.

If then C be the centre of gravity of equal masses placed at the feet of these perpendiculars we have (§ 4)

$$\bar{x}_i=\frac{1}{n+1}\left[\rho_1 l_{1i}+\rho_2 l_{2i}+\dots\dots\rho_{(n+1)}l_{(n+1)i}+(n+1)\bar{x}_i\right]$$

or $$\sum_{j=1}^{j=n+1}\rho_j l_{ji}=0;$$

There are n such relations corresponding to $i=1, 2, \dots\dots n$.

Multiplying these n relations by $l_{j1}, l_{j2}, \ldots\ldots l_{jn}$ respectively and adding, we get

$$\rho_1\left(\sum_{i=1}^{i=n} l_{ji} l_{1i}\right)+\rho_2\left(\sum_{i=1}^{i=n} l_{ji} l_{2i}\right)+\ldots$$

$$+l_j+\ldots\ldots+\rho_{(n+1)}\left(\sum_{i=1}^{i=n} l_{ji} l_{(n+1)i}\right)=0$$

There are $n+1$ such relations corresponding to $j=1, 2, \ldots\ldots n+1$.

Hence we see that $\rho_1, \rho_2, \ldots\ldots \rho_{n+1}$, are proportional to

$$V_1, V_2, \ldots\ldots V_{n+1}.$$

2. Let us consider the linear loci $L^{(r)}-L^{(s)}, \ldots$(I), which are $\frac{1}{2} n(n+1)$ in number. It may be easily seen that there are only n of these loci which are independent, any other may always be found to pass through the points of intersection of two of the n independent. But n independent linear loci (in an n-space) always meet in a point. These loci (I) may be regarded, as in the ordinary geometry, with regard to the conventions we have assumed, as the bisectors of the angles between the faces of the simplex. The $\frac{1}{2} n(n-1)$ bisecting linear loci passing through any vertex intersect one another in a line (for, there are only $n-1$ independent linear equations), and this line is then equidistant from the n faces of the simplex passing through the vertex.* Thus there are $(n+1)$ such lines, each passing through one vertex and these lines meet in a point. This point has been called the centre of the n-spheric inscribed within the simplex.

Solving the n equations (I) for $s=j$, $r=1, 2, \ldots\ldots j-1, j+1, \ldots\ldots n+1$, the co-ordinates of the centre are given by †

$$\frac{x_1}{P_1}=\frac{x_2}{P_2}=\ldots\ldots=\frac{x_n}{P_n}=\frac{1}{P} \text{ (say)}$$

* *Cf.* Manning, *Geometry of four dimensions*, §§ 73, 118.

† An elegant treatment of the solution of any system of linear equations is to be found in Dr. Cullis's *Matrices and Determinoids*. The above solution can be written as

$$[x_1 x_2 \ldots\ldots x_n\ 1]=\rho[P_1 P_2 \ldots\ldots P_n\ P]$$

where ρ is an unspecified scalar quantity. See Vol. I, Chap. XI, § 88.

where

$$P = \pm \begin{vmatrix} l_{11}-l_{j1} & l_{12}-l_{j2} & \cdots & l_{1n}-l_{jn} \\ l_{21}-l_{j1} & l_{22}-l_{j2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{(j-1)1}-l_{j1} & \cdots & \cdots & \cdots \\ l_{(j+1)1}-l_{j1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{(n+1)1}-l_{j1} & \cdots & \cdots & l_{(n+1)n}-l_{jn} \end{vmatrix},$$

according as n is even or odd.

This determinant, on developing in the ordinary way, reduces to the sum of $n+1$ others of the same order

$$= \pm \left[\begin{vmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ l_{(j-1)1} & \cdots & \cdots & \cdots \\ l_{(j+1)1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{(n+1)1} & \cdots & \cdots & l_{(n+1)n} \end{vmatrix} \right.$$

$$- \left\{ \begin{vmatrix} l_{11} & \cdots & l_{1(n-1)} & l_{jn} \\ l_{21} & \cdots & l_{2(n-1)} & l_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ l_{(j-1)1} & \cdots & \cdots & \cdots \\ l_{(j+1)1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{(n+1)1} & \cdots & \cdots & \cdots \end{vmatrix} \right.$$

$$
+\begin{vmatrix}
l_{11} & \cdots & l_{j(n-1)} & l_{1n} \\
l_{21} & \cdots & l_{j(n-1)} & {}_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
l_{(j-1)1} & \cdots & \cdots & \cdots \\
l_{(j+1)1} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
l_{(n+1)1} & \cdots & {}_{j(n-1)} & l_{(n+1)n}
\end{vmatrix} + \cdots\cdots
$$

$$
+\begin{vmatrix}
l_{j1} & l_{12} & \cdots & l_{1n} \\
l_{j1} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & l_{(j-1)2} & \cdots & \cdots \\
\cdots & l_{(j+1)2} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
l_{j1} & \cdots & \cdots & l_{(n+1)n}
\end{vmatrix} \Big\} \Big].
$$

Therefore, if each of the determinants within the second bracket be expanded as functions of equal constituents of the row containing them, we have, as a simplification of the determinant,

$$
+\Bigg[\begin{vmatrix}
l_{21} & l_{22} & \cdots & l_{2n} \\
l_{31} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
l_{(n+1)1} & \cdots & \cdots & l_{(n+1)n}
\end{vmatrix}
$$

$$
-\begin{vmatrix}
l_{11} & l_{12} & \cdots & l_{1n} \\
l_{31} & l_{32} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
l_{(n+1)1} & \cdots & \cdots & l_{(n+1)n}
\end{vmatrix} + \cdots\cdots \Bigg].
$$

So that ultimately

$$P=\sum_{j=1}^{j=n+1} L_{j(n+1)}.$$

Similarly $$P_i=\sum_{j=1}^{j=n+1} L_{ji},$$

and so on for the others.

Hence the co-ordinates of the centre of the n-spheric inscribed within the simplex are given by

$$x_i=\frac{\sum_{j=1}^{j=n+1} L_{ji}}{\sum_{j=1}^{j=n+1} L_{j(n+1)}} \qquad \text{(II)}$$

3. Any linear locus passing through the points of intersection of the faces $L^{(r)}$ and $L^{(s)}$ is of the form

$$\lambda L^{(r)}+\mu L^{(s)},$$

where λ and μ are two arbitraries; if, moreover, this locus passes through the middle point (a_i) of the opposite line, *i.e.*, the line of intersection of the remaining $n-1$ faces of the simplex, its equation reduces to the form

$$L^{(r)} L^{(s)}_{(a_i)} - L^{(s)} L^{(r)}_{(a_i)};$$

where $L^{(r)}_{(a_i)}$ and $L^{(s)}_{(a_i)}$ are the results of substituting a_i for x_i in $L^{(r)}$ and $L^{(s)}$ respectively.

But $$a_i=\tfrac{1}{2}\left[\frac{L_{ri}}{L_{r(n+1)}}+\frac{L_{si}}{L_{s(n+1)}}\right],$$

therefore $$\frac{L^{(r)}_{(a_i)}}{2L_{r(n+1)}\,L_{s(n+1)}}=L_{r(n+1)}\left(\sum_{j=1}^{j=n+1} l_{rj}L_{sj}\right)$$

$$+L_{s(n+1)}\left(\sum_{j=1}^{j=n+1} l_{rj}L_{rj}\right)$$

$$=L_{s(n+1)}\,\Delta \quad (\text{since } \textstyle\sum l_{rj}L_{sj}=0)$$

Hence the locus under consideration may be written as

$$L_{r(n+1)}\, L^{(r)} - L_{s(n+1)}\, L^{(s)} \qquad \text{(III)};$$

there are, as before $\frac{1}{2}n(n+1)$ such loci and they intersect in a point.

Now let us consider the two loci

$$L_{r(n+1)}\, L^{(r)} - L_{s(n+1)}\, L^{(s)}, \; L_{r(n+1)}\, L^{(r)} - L_{p(n+1)}\, L^{(p)}$$

of (III). The first locus passes through the points common to $L^{(r)}$ and $L^{(s)}$, *i.e.*, it passes through the $n-1$ vertices

$$A_1,\; A_2, \ldots\ldots A_{r-1},\; A_{r+1}, \ldots\ldots A_{s-1},\; A_{s+1}, \ldots\ldots A_p, \ldots\ldots A_{n+1}$$

of the simplex, and through the middle point of the line $A_r\, A_s$, *i.e.*, it passes through the $n-2$ vertices,

$$A_1, \ldots\ldots A_{r-1},\; A_{r+1}, \ldots\ldots A_{s-1},\; A_{s+1}, \ldots\ldots A_{p-1},\; A_{p+1}, \ldots\ldots A_{n+1}$$

and through the bisector of the side A_rA_s of the triangle $A_r\, A_s\, A_p$. Similarly the second locus passes through the same $n-2$ vertices and through the bisector of another side A_rA_p of the same triangle. Hence the two loci taken together represent a space of $n-2$ dimensions passing through the $n-2$ vertices and through the centroid of the opposite triangle. Thus, there are

$$\frac{(n+1).n.(n-1)}{3!}$$

independent spaces of $n-2$ dimensions each of which passing through a group of $n-2$ vertices passes through the controid of the opposite triangle and these spaces intersect in a point. In a similar way the three loci

$$L_{r(n+1)}\, L^{(r)} - L_{s(n+1)}\, L^{(s)}, \;\; L_{r(n+1)}\, L^{(r)} - L_{p(n+1)}\, L^{(p)},$$

$$L_{s(n+1)}\, L^{(s)} - L_{q(n+1)}\, L^{(q)}$$

taken together represent a space of $n-3$ dimensions which passing through the $n-3$ vertices, *i.e.*, all the vertices but A_r, A_s, A_p, A_q passes through the centroid of the opposite tetrahedron. There are

$$\frac{(n+1)n(n-1)(n-2)}{4!}$$

such spaces and they intersect in the same point. Proceeding thus we have finally

$$\frac{(n+1)!}{n!}$$

or $n+1$ lines each of which passing through a vertex passes through the centroid of the opposite simplex of $(n-1)$th order.* These lines meet in the same point, given by equations (III , which has been called the centroid of the simplex. (It is apparent that the idea of the centroid of a simplex is the same as that of centre of gravity of equal masses placed at its vertices). Solving n equations of (III) for

$$s=j;\quad r=1, 2, \ldots\ldots j-1, j+1, \ldots\ldots n+1,$$

we have

$$\frac{x_1}{Q_1}=\frac{x_2}{Q_2}=\ldots\ldots\frac{x_n}{Q_n}=\frac{1}{Q}\ (\text{say}),$$

where

$$Q=\pm\left|\begin{array}{llll}
l_{11}L_{1(n+1)}-l_{j1}L_{j(n+1)} & l_{12}L_{1(n+1)}-l_{j2}L_{j(n+1)} & \ldots\ldots & l_{1n}L_{1(n+1)} \\
 & & & \quad -l_{jn}L_{j(n+1)} \\
l_{21}L_{2(n+1)}-l_{j1}L_{j(n+1)} & & \ldots & l_{2n}L_{2(n+1)}-l_{jn}L_{j(n+1)} \\
\ldots & \ldots & \ldots & \ldots \\
l_{(j-1)1}L_{(j-1)(n+1)}-l_{j1}L_{j(n+1)} & & \ldots & \ldots \\
l_{(j+1)1}L_{(j+1)(n+1)}-l_{j1}L_{j(n+1)} & & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
l_{(n+1)1}L_{(n+1)(n+1)}-l_{j1}L_{j(n+1)} & & \ldots & l_{(n+1)n}L_{(n+1)(n+1)} \\
 & & & \quad -l_{jn}L_{j(n+1)}
\end{array}\right|$$

according as n is even or odd.

* *Cf.* Manning, § 118.

This determinant, when expanded in the ordinary way, is equal to the sum of $(n+1)$ others of the same order.

$$
=\pm\Bigg[\begin{vmatrix}
l_{11}L_{1(n+1)} & l_{12}L_{1(n+1)} & \cdots & l_{1n}L_{1(n+1)} \\
\cdots & \cdots & \cdots & \cdots \\
l_{(j-1)1}L_{(j-1)(n+1)} & \cdots & \cdots & \cdots \\
l_{(j+1)1}L_{(j+1)(n+1)} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
l_{(n+1)1}L_{(n+1)(n+1)} & \cdots & \cdots & l_{(n+1)n}L_{(n+1)(n+1)}
\end{vmatrix}
$$

$$
-\Bigg\{\begin{vmatrix}
l_{11}L_{1(n+1)} & l_{12}L_{1(n+1)} & \cdots & l_{jn}L_{j(n+1)} \\
\cdots & \cdots & \cdots & l_{jn}L_{j(n+1)} \\
\cdots & \cdots & \cdots & \cdots \\
l_{(j-1)1}L_{(j-1)(n+1)} & & \cdots & \cdots \\
l_{(j+1)1}L_{(j+1)(n+1)} & & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
l_{(n+1)1}L_{(n+1)(n+1)} & & \cdots & l_{jn}L_{j(n+1)}
\end{vmatrix}
$$

$$
+\cdots
$$

$$
+\begin{vmatrix}
l_{j1}L_{j(n+1)} & l_{12}L_{1(n+1)} & \cdots & l_{1n}L_{1(n+1)} \\
l_{j1}L_{j(n+1)} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & l_{(j-1)2}L_{(j-1)(n+1)} & & \cdots \\
\cdots & l_{(j+1)2}L_{(j+1)(n+1)} & & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
l_{j1}L_{j(n+1)} & \cdots & \cdots & l_{(n+1)n}L_{(n+1)(n+1)}
\end{vmatrix}\Bigg\}\Bigg].
$$

This when simplified is reduced, as before, to

$$=\pm\left[\begin{vmatrix} l_{21}L_{2(n+1)} & \cdots & l_{2n}L_{2(n+1)} \\ l_{31}L_{3(n+1)} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ l_{(n+1)1}L_{(n+1)(n+1)} & \cdots & l_{(n+1)n}L_{(n+1)(n+1)} \end{vmatrix}\right.$$

$$\left.-\begin{vmatrix} l_{11}L_{1(n+1)} & \cdots & l_{1n}L_{1(n+1)} \\ l_{31}L_{3(n+1)} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ l_{(n+1)1}L_{(n+1)(n+1)} \cdots & & l_{(n+1)n}L_{(n+1)(n+1)} \end{vmatrix}+\cdots\right].$$

So that

$$Q=(n+1)\Pi L_{j(n+1)}, \qquad j=1, 2, \ldots\ldots n+1.$$

Again if in the determinant Q we replace

$$l_{r1}, l_{r2}, \ldots\ldots l_{rn} \quad \text{by} \quad l_{r2}, l_{r3}, \ldots\ldots l_{r(n+1)} \quad \text{respectively,}$$

$$r=1, 2, \ldots\ldots n+1,$$

and so on, we have

$$Q_1=\begin{vmatrix} l_{22}L_{2(n+1)} & \cdots & l_{2(n+1)}L_{2(n+1)} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ l_{(n+1)2}L_{(n+1)(n+1)} \cdots & & l_{(n+1)(n+1)}L_{(n+1)(n+1)} \end{vmatrix} \ldots+\ldots$$

$$=L_{11}L_{2(n+1)}L_{3(n+1)}\ldots\ldots L_{(n+1)(n+1)}$$

$$+L_{21}L_{1(n+1)}L_{3(n+1)}\ldots\ldots L_{(n+1)(n+1)}+\ldots\ldots$$

$$+L_{(n+1)1}L_{1(n+1)}L_{2(n+1)}\ldots\ldots L_{n(n+1)}.$$

Thus the co-ordinates of the centroid are given by

$$x_i=\frac{1}{n+1}\left[\sum_{j=1}^{j=n+1}\frac{L_{ji}}{L_{j(n+1)}}\right], \quad i=1, 2, \ldots n. \qquad \text{(IV).}$$

4. Any linear locus passing through the vertex A_j of the simplex is of the form

$$\lambda_1 L^{(1)} + \lambda_2 L^{(2)} + \ldots\ldots \lambda_{j-1} L^{(j-1)} + \lambda_{j+1} L^{(j+1)} + \ldots\ldots + \lambda_{n+1} L^{(n+1)},$$

where $\lambda_1, \lambda_2 \ldots\ldots$ are any n arbitrary quantities. If it is, moreover, parallel to the opposite face $L^{(j)}$, then, since the relations

$$\frac{\lambda_1 l_{11} + \ldots\ldots + \lambda_{j-1} l_{(j-1)1} + \lambda_{j+1} l_{(j+1)1} + \ldots\ldots + \lambda_{n+1} l_{(n+1)1}}{l_{j1}} = \text{etc}$$

will determine the ratios of $\lambda_1, \lambda_2, \ldots$ to one another, the equation is easily reduced to the form

$$L_{1(n+1)} L^{(1)} + \ldots + L_{(j-1)(n+1)} L^{(j-1)}$$
$$+ L_{(j+1)(n+1)} L^{(j+1)} + \ldots + L_{(n+1)(n+1)} L^{(n+1)} .$$

There are $n+1$ such loci corresponding to

$$j = 1, 2, \ldots\ldots n+1,$$

and these loci form a second simplex of the nth order which is, evidently, similar to the original one. It is easy to see that the centroid of this derived simplex coincides with that of the original one. For, since

$$l_{1j} L_{1(n+1)} + l_{2j} L_{2(n+1)} + \ldots + l_{(n+1)j} L_{(n+1)(n+1)}$$

vanishes for $j = 1, 2, \ldots n$ and is equal to Δ for $j = n+1$, the above equations may be written as

$$L_{j(n+1)} L^{(j)} = \Delta \qquad \text{(V)}.$$

Solving the equations (v) for $j = 1, 2, \ldots n$, we get

$$\frac{x_1}{R_1} = \frac{x_2}{R_2} = \ldots = \frac{x_n}{R_n} = \frac{1}{R} \text{ (say)},$$

where

$$R \equiv \pm \begin{vmatrix} l_{11}L_{1(n+1)} & \cdots & l_{1n}L_{1(n+1)} \\ l_{21}L_{2(n+1)} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ l_{n1}L_{n(n+1)} & \cdots & l_{nn}L_{n(n+1)} \end{vmatrix}$$

according as n is even or odd

$$= \Pi L_{j(n+1)}, \quad j=1, 2, \ldots n+1, \quad \text{and}$$

$$R_1 \equiv \begin{vmatrix} l_{12}L_{1(n+1)} & l_{13}L_{1(n+1)} & \cdots & l_{1(n+1)}L_{1(n+1)} - \Delta \\ l_{22}L_{2(n+1)} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{n2}L_{n(n+1)} & \cdots & \cdots & l_{n(n+1)}L_{n(n+1)} - \Delta \end{vmatrix}$$

$$= L_{(n+1)1} L_{1(n+1)} L_{2(n+1)} \ldots L_{n(n+1)} - \Delta L_{1(n+1)} \cdots L_{n(n+1)}$$

$$\times \begin{vmatrix} l_{12} & \cdots & l_{1n} & \dfrac{1}{L_{1(n+1)}} \\ l_{22} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{n2} & \cdots & l_{nn} & \dfrac{1}{L_{n(n+1)}} \end{vmatrix}.$$

But the last determinant of the nth order

$$= \mp \Bigg[\frac{1}{L_{1(n+1)}} \begin{vmatrix} l_{22} & \cdots & \cdots & l_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{n2} & \cdots & \cdots & l_{nn} \end{vmatrix}$$

$$-\frac{1}{L_{2(n+1)}}\begin{vmatrix} l_{12} & \cdots & \cdots & l_{1n} \\ l_{32} & \cdots & \cdots & l_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ l_{n2} & \cdots & \cdots & l_{nn} \end{vmatrix} + \cdots \Big],$$

according as n is even or odd.

$$= \frac{-1}{\Delta}\Big[\frac{1}{L_{1(n+1)}}\begin{vmatrix} L_{11} & L_{1(n+1)} \\ L_{(n+1)1} & L_{(n+1)(n+1)} \end{vmatrix}$$

$$+\frac{1}{L_{2(n+1)}}\begin{vmatrix} L_{21} & L_{2(n+1)} \\ L_{(n+1)1} & L_{(n+1)(n+1)} \end{vmatrix} + \cdots \Big]$$

$$\therefore \quad R_1 = L_{(n+1)1} L_{1(n+1)} L_{2(n+1)} \cdots L_{n(n+1)}$$

$$+ L_{11} L_{2(n+1)} L_{3(n+1)} \cdots L_{(n+1)(n+1)}$$

$$+ L_{21} L_{1(n+1)} L_{3(n+1)} \cdots L_{(n+1)(n+1)} + \cdots$$

$$+ L_{n1} L_{1(n+1)} L_{2(n+1)} \cdots L_{(n-1)(n+1)} L_{(n+1)(n+1)}$$

$$- n L_{(n+1)1} L_{1(n+1)} L_{2(n+1)} \cdots L_{n(n+1)}.$$

Thus the co-ordinates of the vertex A'_j of the derived simplex are given by

$$x_i = \frac{-(n-1)L_{ji}}{L_{j(n+1)}} + \Sigma \frac{L_{ri}}{L_{r(n+1)}} \qquad \text{(VI)},$$

where Σ is the summation for

$$r = 1, 2, \ldots j-1, j+1, \ldots n+1;$$

and the n co-ordinates are for $i = 1, 2, \ldots n$. Thus we see that the two centroids coincide.

Again it follows that the co-ordinates of the centroid of the simplex of the $(n-1)$th order whose vertices are all the vertices but A'_j of the derived simplex are given by

$$x_i = \frac{1}{n}\left[\left\{ \frac{-(n-1)\,L_{1i}}{L_{1(n+1)}} + \frac{L_{2i}}{L_{2(n+1)}} + \ldots + \frac{L_{(n+1)i}}{L_{(n+1)(n+1)}} \right\}\right.$$

$$+\left\{\frac{L_{1i}}{L_{1(n+1)}} - \frac{(n-1)\,L_{2i}}{L_{2(n+1)}} + \ldots\right\}$$

$$+\ldots+\left\{\frac{L_{1i}}{L_{1(n+1)}} + \ldots - \frac{(n-1)\,L_{(j-1)i}}{L_{(j-1)(n+1)}} + \ldots\right\}$$

$$+\left\{\frac{L_{1i}}{L_{1(n+1)}} + \ldots - \frac{(n-1)L_{(j+1)i}}{L_{(j+1)(n+1)}} + \ldots\right\}$$

$$\left. + \ldots\right]$$

$$= \frac{L_{ji}}{L_{j(n+1)}},$$

which is the x_i co-ordinate of the vertex A_j of the original simplex. Thus the original simplex has its vertices at the centroids of the faces of the derived simplex.

Again the centroid of the surface-content of the derived simplex is the centre of the n-spheric inscribed in the original simplex †, and are thus given by (II).

The linear locus bisecting the angle between the faces

$$L_{j(n+1)}\,L^{(j)} - \Delta \quad \text{and} \quad L_{r(n+1)}\,L^{(r)} - \Delta$$

of the derived simplex is easily seen to be

$$L^{(r)} - L^{(j)} - \Delta\cdot\left(\frac{1}{L_{r(n+1)}} - \frac{1}{L_{j(n+1)}}\right).$$

There are $\frac{n(n+1)}{2}$ of these equations of which, as before, only n are independent and thus these loci meet in a point. Solving these

* *Vide*, Burnside and Panton, *Theory of Equations*, Vol. II, § 146.

† *Proceedings of the London Math. Soc.*, 'Properties of Simplicissima,' Vol XVIII, § XV.

equations for $r=1, 2, \cdots, j-1, j+1, \ldots n+1$ we get as the co-ordinates of the centre of the n-spheric inscribed within the simplex

$$\frac{x_1}{S_1}=\frac{x_2}{S_2}=\ldots=\frac{x_n}{S_n}=\frac{1}{S}, \quad \text{where}$$

$$S=+\begin{vmatrix} l_{11}-l_{j1} & \cdots & \cdots & l_{1n}-l_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ l_{(j-1)1}-l_{j1} & \cdots & \cdots & \cdots \\ l_{(j+1)1}-l_{j1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{(n+1)1}-l_{j1} & \cdots & \cdots & l_{(n+1)n}-l_{jn} \end{vmatrix},$$

according as n is even or odd,

$$=\sum_{j=1}^{j=n+1} L_{j(n+1)}$$

$$S_1=\begin{vmatrix} l_{12}-l_{j2} & \cdots & l_{1(n+1)}-l_{j(n+1)} \\ \cdots & \cdots & \cdots \\ l_{(j-1)2}-l_{j2} & \cdots & \cdots \\ l_{(j+1)2}-l_{j2} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ l_{(n+1)2}-l_{j2} & \cdots & l_{(n+1)(n+1)}-l_{j(n+1)} \end{vmatrix}$$

$$-\Delta.\begin{vmatrix} l_{12}-l_{j2} & \cdots & l_{1n}-l_{jn} & \frac{1}{L_{1(n+1)}}-\frac{1}{L_{j(n+1)}} \\ \cdots & \cdots & \cdots & \cdots \\ l_{(j-1)2}-l_{j2}\cdots & & \cdots & \cdots \\ l_{(j+1)2}-l_{j2}\cdots & & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{(n+1)2}-l_{j2}\cdots & & l_{(n+1)n}-l_{jn} & \frac{1}{L_{(n+1)(n+1)}}-\frac{1}{L_{j(n+1)}} \end{vmatrix}$$

The first determinant

$$= \sum_{j=1}^{j=n+1} L_{j1},$$

and the co-efficient of $-\Delta$ differs from this determinant only in the last column, where in place of $l_{1(n+1)}-l_{j(n+1)},\ldots$ we have

$$\frac{1}{L_{1(n+1)}}-\frac{1}{L_{j(n+1)}},\ldots;$$

so that it may be written as the sum of $n+1$ determinants

$$\begin{vmatrix} l_{22} & l_{23} & \cdots & \dfrac{1}{L_{2(n+1)}} \\ l_{32} & l_{33} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{(n+1)2} & \cdots & \cdots & \dfrac{1}{L_{(n+1)(n+1)}} \end{vmatrix}$$

$$-\begin{vmatrix} l_{12} & l_{13} & \cdots & \dfrac{1}{L_{1(n+1)}} \\ l_{32} & l_{33} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ l_{(n+1)2} & \cdots & \cdots & \dfrac{1}{L_{(n+1)(n+2)}} \end{vmatrix}+\ldots$$

$$=\frac{1}{\Delta}\left[\left\{\frac{1}{L_{2(n+1)}}\begin{vmatrix} L_{11} & L_{1(n+1)} \\ L_{21} & L_{2(n+1)} \end{vmatrix}+\ldots\right.\right.$$

$$\left.+\frac{1}{L_{(n+1)(n+1)}}\begin{vmatrix} L_{11} & L_{1(n+1)} \\ L_{n1} & L_{n(n+1)} \end{vmatrix}\right\}$$

$$+\left\{\frac{1}{L_{1(n+1)}}\begin{vmatrix} L_{21} & L_{2(n+1)} \\ L_{11} & L_{1(n+1)} \end{vmatrix}\right.$$

$$\left.\left.+\frac{1}{L_{3(n+1)}}\begin{vmatrix} L_{21} & L_{2(n+1)} \\ L_{31} & L_{3(n+1)} \end{vmatrix}+\ldots\right\}+\ldots\right]$$

$$=\frac{1}{\Delta}\Big[n(L_{11}+L_{21}+\ldots+L_{(n+1)1})$$

$$-\frac{L_{11}}{L_{1(n+1)}}(L_{2(n+1)}+L_{3(n+1)}+\ldots+L_{(n+1)(n+1)})$$

$$-\frac{L_{21}}{L_{2(n+1)}}(L_{1(n+1)}+L_{3(n+1)}+\ldots+L_{(n+1)(n+1)})\ldots$$

$$-\frac{L_{(n+1)1}}{L_{(n+1)(n+1)}}(L_{1(n+1)}+\ldots+L_{n(n+1)})\Big]$$

$$=\frac{1}{\Delta}\Big[n(L_{11}+L_{21}+\ldots+L_{(n+1)1})-\Big\{\frac{L_{11}}{L_{1(n+1)}}(L_{1(n+1)}$$

$$+L_{2(n+1)}+\ldots+L_{(n+1)(n+1)})-L_{11}\Big\}-\ldots\Big],$$

$$=\frac{1}{\Delta}\Big[(n+1)\Big(\sum_{j=1}^{j=n+1}L_{j1}\Big)$$

$$-\Big(\sum_{j=1}^{j=n+1}L_{j(n+1)}\Big)\Big(\sum_{j=1}^{j=n+1}\frac{L_{j1}}{L_{j(n+1)}}\Big)\Big].$$

$$\therefore\quad S_1=\Big(\sum_{j=1}^{j=n+1}L_{j(n+1)}\Big)\Big(\sum_{j=1}^{j=n+1}\frac{L_{j1}}{L_{j(n+1)}}\Big)$$

$$-n\Big(\sum_{j=1}^{j=n+1}L_{j1}\Big).$$

Thus the co-ordinates of the centre of the n-spheric inscribed within the derived simplex are given by

$$x_i=\sum_{j=1}^{j=n+1}\frac{L_{ji}}{L_{j(n+1)}}-n\frac{\sum_{j=1}^{j=n+1}L_{ji}}{\sum_{j=1}^{j=n+1}L_{j(n+1)}} \tag{VII}$$

Now suppose O, G, H are respectively the centre of the inscribed n-spheric, the centroid and the centroid of the surface-content of the derived simplex.

The line joining O and G are given by

$$\frac{x_1 - \left[\Sigma \dfrac{L_{j1}}{L_{j(n+1)}} - n.\dfrac{\Sigma L_{j1}}{\Sigma L_{j(n+1)}}\right]}{\dfrac{1}{n+1}\left(\Sigma \dfrac{L_{j1}}{L_{j(n+1)}}\right) - \left[\Sigma \dfrac{L_{j1}}{L_{j(n+1)}} - n\,\dfrac{\Sigma L_{j1}}{\Sigma L_{j(n+1)}}\right]} = \ldots = \rho \text{ (say)}$$

whence

$$x_i = n\rho\left[\frac{\Sigma L_{ji}}{\Sigma L_{j(n+1)}} - \frac{1}{n+1}\left(\Sigma \frac{L_{ji}}{L_{j(n+1)}}\right)\right]$$

$$+\Sigma \frac{L_{ji}}{L_{j(n+1)}} - n\,\frac{\Sigma L_{ji}}{\Sigma L_{j(n+1)}}.$$

(Σ always extends from $j=1$ to $j=n+1$)

Hence if

$$\rho = \frac{n+1}{n}, \qquad x_i = \frac{\Sigma L_{ji}}{\Sigma L_{j(n+1)}},$$

which is the x_i co-ordinate of H.

Thus OG passes through H and OG$=n$.GH.

5. The equation of an n-spheric may always be put into the form

$$\sum_{i=1}^{i=n} x_i^2 + 2\sum_{i=1}^{i=n} B_i x_i + C = 0,$$

where B_i and C are $n+1$ arbitrary constants.

If it passes through the $n+1$ vertices of the simplex we shall have

$$\frac{\sum\limits_{i=1}^{i=n} L_{ji}^2}{L_{j(n+1)}^2} + 2.\frac{\sum\limits_{i=1}^{i=n} B_i L_{ji}}{L_{j(n+1)}} + C = 0,$$

there being $n+1$ such relations

corresponding to $j=1,2,\ldots\ldots n+1$. Eliminating these arbitraries between these relations and the equation we get as the equation of the n-spheric circumscribed about the simplex

$$\begin{vmatrix} \Sigma x_i^2 & x_1 & x_2 & \ldots & x_n & & 1 \\ \dfrac{\Sigma L^2_{1i}}{L_{1(n+1)}} & L_{11} & L_{12} & \ldots & L_{1n} & \ldots & L_{1(n+1)} \\ \ldots & \ldots & & \ldots & & \ldots & \ldots \\ \ldots & \ldots & & \ldots & & \ldots & \ldots \\ \dfrac{\Sigma L^2_{(n+1)i}}{L_{(n+1)(n+1)}} & \ldots & & \ldots & & \ldots & L_{(n+1)(n+1)} \end{vmatrix}$$

$=0$, where Σ extends from $i=1$ to $i=n$.

Or $F(\Sigma x_i^2)+2\Sigma G_i x_i+H=0$,

where $F=\Delta^n$,

$$-2G_i=\pm\begin{vmatrix} \dfrac{\Sigma L^2_{1i}}{L_{1(n+1)}} & L_{11} & L_{1(i-1)} & L_{1(i+1)} & L_{1(n+1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \dfrac{\Sigma L^2_{(n+1)i}}{L_{(n+1)(n+1)}} & \ldots & \ldots & \ldots & L_{(n+1)(n+1)} \end{vmatrix}$$

according as i is odd or even.

$$=\pm\left[\frac{\Sigma L^2_{1i}}{L_{1(n+1)}}\begin{vmatrix} L_{21} & L_{2(i-1)} & L_{2(i+1)} & L_{2(n+1)} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ L_{(n+1)1} & \ldots & \ldots & L_{(n+1)(n+1)} \end{vmatrix}\ldots\right]$$

$$=\Delta^{n-1}\left[\frac{l_{1i}(\Sigma L^2_{1i})}{L_{1(n+1)}}+\frac{l_{2i}(\Sigma L^2_{2i})}{L_{2(n+1)}}+\quad\ldots\quad\ldots\right.$$

$$\left.+\frac{l_{(n-1)i}(\Sigma L^2_{(n+1)i})}{L_{(n+1)(n+1)}}\right].$$

Similarly

$$-\mathrm{H} = \Delta^{n-1}\left[\frac{l_{1(n+1)}(\Sigma \mathrm{L}^2{}_{1i})}{\mathrm{L}_{1(n+1)}} + \ldots + \frac{l_{(n+1)(n+1)}\,(\Sigma \mathrm{L}^2{}_{(n+1)\,i})}{\mathrm{L}_{(n+1)(n+1)}}\right]$$

The line joining the centre of the n-spheric circumscribed about the simplex and the vertex Aj is given by

$$\frac{x_1 - \dfrac{\mathrm{L}_{j1}}{\mathrm{L}_{j(n+1)}}}{\dfrac{1}{2\Delta}\left[\dfrac{l_{11}\,(\Sigma \mathrm{L}^2{}_{1i})}{\mathrm{L}_{1(n+1)}} + \dfrac{l_{21}\,(\Sigma \mathrm{L}^2{}_{2i})}{\mathrm{L}_{2(n+1)}} + \ldots \ldots + \dfrac{l_{(n+1)1}(\Sigma \mathrm{L}^2{}_{(n+1)\,i})}{\mathrm{L}_{(n+1)(n+1)}}\right] - \dfrac{\mathrm{L}_{j1}}{\mathrm{L}_{j(n+1)}}} = \ldots \quad \ldots$$

If, therefore, ρ_j represent the length of this line joining A_j to the opposite face $\mathrm{L}^{(j)}$ passing through this centre, then since the square-root of the sum of the squares of the denominators of the above equal ratios is exactly equal to the radius, R, of the n-spheric, each of the ratios must be equal to $\frac{\rho_j}{\mathrm{R}}$; then ρ_j is given by

$$\frac{\rho_j}{2\Delta\mathrm{R}}\left[l_{j1}\left\{\frac{l_{11}\,(\Sigma\,\mathrm{L}^2_{1i})}{\mathrm{L}_{1(n+1)}} + \ldots\ldots\ldots\right\}\right.$$

$$+\, l_{j2}\left\{\frac{l_{12}\,(\Sigma\,\mathrm{L}^2_{1i})}{\mathrm{L}_{1(n+1)}} + \ldots\ldots\ldots\right\} + \ldots\ldots\ldots\ldots\ldots$$

$$\left. - \frac{2\,\Delta \Sigma\, l_{ji}\mathrm{L}_{ji}}{\mathrm{L}_{j(n+1)}}\right] + \frac{\Sigma l_{ji}\mathrm{L}_{ji}}{\mathrm{L}_{j(n+1)}} + l_{j\,(n+1)} = 0, \text{ by virtue of}$$

the equation to $\mathrm{L}^{(j)}$.

Or

$$\frac{\rho_j}{2\Delta\mathrm{R}}\left[\left\{\frac{\Sigma\,\mathrm{L}^2_{1i}}{\mathrm{L}_{1(n+1)}}\left(\Sigma l_{ji}\ l_{1i}\right) + \frac{\Sigma\,\mathrm{L}^2_{2i}}{\mathrm{L}_{2(n+1)}}\left(\Sigma l_{ji}\ l_{2i}\right) + \ldots\right.\right.$$

$$+\frac{\Sigma L_{ji}^2}{L_{j(n+1)}}+\ldots\ldots\ldots+\frac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}}\left(\Sigma l_{ji}\ l_{(n+1)i}\right)\Bigg\}$$

$$+\frac{2\,\Delta\,(l_{j(n+1)}\ L_{j(n+1)}-\Delta}{L_{j(n+1)}}\Bigg]+\frac{\Delta}{L_{j(n+1)}}=0,$$

Σ extending from $i=1$ to $i=n$.

There are $n+1$ such lengths corresponding to $j=1,2\ldots\ldots\ldots n+1$.

Whence

$$\sum_{j=1}^{j=n+1}\frac{R}{\rho_j}=\frac{-1}{2\Delta^2}\Big[(\Sigma L_{1i}^2+\Sigma L_{2i}^2+\ldots\ldots\ldots+\Sigma L_{(n+1)i}^2)$$

$$+\frac{\Sigma L_{1i}^2}{L_{1(n+1)}}\Big\{L_{2(n+1)}(\Sigma l_{1i}\ l_{2i})+L_{3(n+1)}(\Sigma l_{1i}\ l_{3i})+\ldots\ldots$$

$$+L_{(n+1)(n+1)}(\Sigma l_{1i}\,l_{(n+1)i})\Big\}+\ldots\ldots$$

$$+\frac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}}\Big\{L_{1(n+1)}(\Sigma l_{(n+1)i}\ l_{1i})+\ldots\ldots\ldots\ldots$$

$$+L_{n(n+1)}(\Sigma l_{(n+1)i}\ l_{ni})\Big\}-2n\Delta^2\Big]$$

But

$$\frac{\Sigma L_{ji}^2}{L_{j(n+1)}}\ [L_{1(n+1)}(\Sigma l_{ji}\cdot l_{1i})+L_{2(n+1)}(\Sigma l_{ji}\ l_{2i})+\ldots$$

$$+L_{(j-1)(n+1)}(\Sigma l_{ji}\ l_{(j-1)i})+L_{(j+1)(n+1)}(\Sigma l_{ji}\ l_{(j+1)i})+\ldots$$

$$+L_{(n+1)(n+1)}(\Sigma l_{ji}\ l_{(n+1)i})]$$

$$= \frac{-\Sigma L_{ji}^2}{L_{j(n+1)}} [l_{j1}(l_{j1} l_{j(n+1)}) + l_{j2}(l_{j2} L_{j(n+1)}) + \ldots\ldots + l_{jn}(l_{jn} l_{j(n+1)})]$$

$= -\Sigma L_{ji}^2$, where in the summation i extends from 1 to n.

Hence

$$\sum_{j=1}^{j=n+1} \frac{R}{\rho_j} = n.$$

Cor. If ρ_j represent the length of the perpendicular from A_j on the opposite face of the simplex, then the coordinates of the foot of the perpendicular are given by

$$x_i = \rho_j l_{ji} + \frac{L_{ji}}{L_{j(n+1)}}.$$

Hence ρ_j is given by

$$\sum_{i=1}^{i=n} l_{ji}(\rho_j l_{ji} + \frac{L_{ji}}{L_{j(n+1)}}) + l_{j(n+1)} = 0,$$ by virtue of the equation to $L^{(j)}$.

By easy simplification we get

$$l_j + \frac{\Delta}{L_{j(n+1)}} = 0$$

Now, if r denote the radius of the n-spheric inscribed within the simplex then, from § 3 (II), r is given by

$$\sum_{i=1}^{i=n} l_{ji} \left[r l_{ji} + \frac{\sum_{j=1}^{j=n+1} L_{ji}}{\sum_{j=1}^{j=n+1} L_{j(n+1)}} \right] + l_{j(n+1)} = 0$$

Or $$r - \frac{l_{j(n+1)} \Sigma L_{j(n+1)} + \Delta}{\Sigma L_{j(n+1)}} + l_{j(n+1)} = 0$$

Or $$r + \frac{\Delta}{\Sigma L_{j(n-1)}} = 0.$$ Thus $$\frac{1}{r} = \sum_{j=1}^{j=n+1} \frac{1}{\rho_j}$$

Hence, when the simplex is regular

$$r=\frac{\rho}{n+1}\,;\ R=\frac{n\rho}{n+1}, \text{ where } \rho \text{ is the altitude}$$

and, from § 2 (i). Sec $\widehat{LL}=n$, where $\widehat{LL}$ is the angle between two faces.

6. From the last article, we have

$$\Sigma G_i^2 = \left[\frac{\Pi L_{j(n+1)}}{2}\right]^2 \times$$

$$\Sigma\begin{vmatrix} \frac{\Sigma L_{1i}^2}{L^2_{1(n+1)}} & \frac{L_{11}}{L_{1(n+1)}} & \frac{L_{12}}{L_{1(n+1)}} & \cdots\cdots & \frac{L_{1(i-1)}}{L_{1(n+1)}} & \frac{L_{1(i+1)}}{L_{1(n+1)}} & \cdots\cdots & \frac{L_{1n}}{L_{1(n+1)}} & 1 \\ \frac{\Sigma L^2_{2i}}{L^2_{2(n+1)}} & \frac{L_{2i}}{L_{2(n+1)}} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \frac{\Sigma L^2_{(n+1)i}}{L^2_{(n+1)(n+1)}} & \cdots & \cdots & \cdots & \cdots & \cdots & & \frac{L_{(n+1)n}}{L_{(n+1)(n+1)}} & 1 \end{vmatrix}$$

$$=\left[\frac{\Pi L_{j(n+1)}}{2}\right]^2 \Sigma\begin{vmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ \frac{\Sigma L^2_{1i}}{L^2_{1(n+1)}} & \frac{L_{11}}{L_{1(n+1)}} & \cdots & \frac{L_{1i}}{L_{1(n+1)}} & \cdots & \frac{L_{1n}}{L_{1(n+1)}} & 1 \\ \frac{\Sigma L^2_{2i}}{L_{2(n+1)}} & \frac{L_{21}}{L_{2(n+1)}} & & \cdots & & \cdots & \cdots \\ \cdots & \cdots & & \cdots & & \cdots & \cdots \\ \frac{\Sigma L^2_{(n+1)i}}{L^2_{(n+1)(n+1)}} & & \cdots & & & \frac{L_{(n+1)n}}{L_{(n+1)(n+1)}} & 1 \end{vmatrix}$$

where Σ always extends from $i=1$ to $i=n$.

$$= \frac{(-1)^{n+1}}{2^n} \left[\frac{\Pi L_{j(n+1)}}{2} \right]^2 \times$$

$$\Sigma \begin{vmatrix} 0 & 0 & \ldots & 1 & \ldots & 0 & 0 \\ \frac{\Sigma L_{1i}^2}{L_{1(n+1)}^2} & \frac{L_{11}}{L_{1(1+1)}} & \ldots & \frac{L_{1i}}{L_{1(n+1)}} & \ldots & \frac{L_{1n}}{L_{1(n+1)}} & 1 \\ \frac{\Sigma L_{2i}^2}{L_{2(n+1)}^2} & \frac{L_{21}}{L_{2(n+1)}} & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}^2} & \ldots & \ldots & \ldots & \ldots & \frac{L_{(n+1)n}}{L_{(n+1)(n+1)}} & 1 \end{vmatrix}$$

$$\times \begin{vmatrix} 0 & 0 & 0 & \ldots & -2 & \ldots & 0 & 0 \\ 1 & \frac{-2L_{11}}{L_{1(n+1)}} & \frac{-2L_{12}}{L_{1(n+1)}} & \ldots & \frac{-2L_{1i}}{L_{1(n+1)}} & \ldots & \frac{-2L_{1n}}{L_{1(n+1)}} & \frac{\Sigma L_{1i}^2}{L_{1(n+1)}^2} \\ 1 & \frac{-2L_{21}}{L_{2(n+1)}} & \frac{-2L_{22}}{L_{2(n+1)}} & \ldots & \ldots & \ldots & \ldots & \frac{\Sigma L_{2i}^2}{L_{2(n+1)}^2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \frac{-2L_{(n+1)1}}{L_{(n+1)(n+1)}} & \ldots & \ldots & \ldots & \ldots & \ldots & \frac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}^2} \end{vmatrix}$$

$$= \frac{(-1)^{n+1}}{2^n}\left[\frac{\Pi L_{j(n+1)}}{2}\right]^2 \times$$

$$\Sigma \begin{vmatrix} -2 & \frac{-2L_{1i}}{L_{1(n+1)}} & \frac{-2L_{2i}}{L_{2(n+1)}} & \cdots & \cdots & \frac{-2L_{(n+1)i}}{L_{(n+1)(n+1)}} \\ \frac{-2L_{1i}}{L_{1(n+1)}} & 0 & (A_1A_2)^2 & (A_1A_3)^2 & \cdots & (A_1A_{n+1})^2 \\ \frac{-2L_{2i}}{L_{2(n+1)}} & (A_2A_1)^2 & 0 & (A_2A_3)^2 & \cdots & (A_2A_{n+1})^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{-2L_{(n+1)i}}{L_{(n+1)(n+1)}} & (A_{n+1}A_1)^2 & \cdots & \cdots & \cdots & 0 \end{vmatrix}$$

Again $-H.F=$

$$(\Pi L_{j(n+1)})^2 \begin{vmatrix} \frac{\Sigma L_{1i}^2}{L_{1(n+1)}^2} & \frac{L_{11}}{L_{1(n+1)}} & \cdots & \frac{L_{1n}}{L_{1(n-1)}} \\ \frac{\Sigma L_{2i}^2}{L_{2(n+1)}^2} & \frac{L_{21}}{L_{2(n+1)}} & \cdots & \frac{L_{2n}}{L_{2(n+1)}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}^2} & \cdots & \cdots & \frac{L_{(n+1)n}}{L_{(n+1)(n+1)}} \end{vmatrix}$$

$\cdots$

$$
\times \begin{vmatrix}
1 & \dfrac{L_{11}}{L_{1(n+1)}} & \dfrac{L_{12}}{L_{1(n+1)}} & \cdots & \dfrac{L_{1n}}{L_{1(n+1)}} \\
1 & \dfrac{L_{21}}{L_{2(n+1)}} & \dfrac{L_{22}}{L_{2(n+1)}} & \cdots & \dfrac{L_{2n}}{L_{2(n+1)}} \\
 & \cdots & \cdots & \cdots & \cdots \\
1 & \dfrac{L_{(n+1)1}}{L_{(n+1)(n+1)}} & \cdots & \cdots & \dfrac{L_{(n+1)n}}{L_{(n+1)(n+1)}}
\end{vmatrix}
$$

$$
= \frac{(\Pi L_{j(n+1)})^2}{(-2)^n}
$$

$$
\times \begin{vmatrix}
0 & 0 & \cdots & 0 & 1 \\
\dfrac{\Sigma L_{1i}^2}{L_{1(n+1)}^2} & \dfrac{L_{11}}{L_{1(n+1)}} & \cdots & \dfrac{L_{1n}}{L_{1(n+1)}} & 1 \\
\dfrac{\Sigma L_{2i}^2}{L_{2(n+1)}^2} & \dfrac{L_{21}}{L_{2(n+1)}} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\dfrac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}^2} & \cdots & \cdots & \dfrac{L_{(n+1)n}}{L_{(n+1)(n+1)}} & 1
\end{vmatrix}
$$

$$
\times \begin{vmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & \dfrac{-2L_{11}}{L_{1(n+1)}} & -\dfrac{2L_{12}}{L_{1(n+1)}} & \cdots & -\dfrac{2L_{1n}}{L_{1(n+1)}} & \dfrac{\Sigma L_{1i}^2}{L_{1(n+1)}^2} \\
1 & -\dfrac{2L_{21}}{L_{2(n+1)}} & \cdots & \cdots & \cdots & \dfrac{\Sigma L_{2i}^2}{L_{2(n+1)}^2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & -\dfrac{2L_{(n+1)1}}{L_{(n+1)(n+1)}} & \cdots & \cdots & \dfrac{-2L_{(n+1)n}}{L_{(n+1)(n+1)}} & \dfrac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}^2}
\end{vmatrix}
$$

$$= \frac{(\Pi L_{j(n+1)})^2}{(-2)^n} \cdot \cdot$$

$$\begin{vmatrix} 1 & \frac{\Sigma L_{1i}^2}{L_{1(n+1)}^2} & \frac{\Sigma L_{2i}^2}{L_{2(n+1)}^2} & \cdots & & \frac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}^2} \\ 1 & 0 & (A_1A_2)^2 & (A_1A_3)^2 & \cdots & (A_1A_{n+1})^2 \\ 1 & (A_2A_1)^2 & 0 & (A_2A_3)^2 & \cdots & (A_2A_{n+1})^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & (A_{n+1}A_1)^2 & \cdots & \cdots & (A_{n+1}A_n)^2 & 0 \end{vmatrix}$$

Hence

$$R^2 = \frac{\Sigma G_i^2}{F^2} - \frac{H}{F} = \frac{1}{F^2} \left(\Sigma G_i^2 - HF \right)$$

$$= \frac{(-1)^n}{2^{n+1}} \left[\frac{\Pi L_{j(n+1)}}{\Delta^n} \right]^2$$

$$\times \left[\Sigma \begin{vmatrix} 1 & \frac{-2L_{1i}}{L_{1(n+1)}} & \frac{-2L_{2i}}{L_{2(n+1)}} & \cdots & -2L_{(n+1)i} \\ \frac{L_{1i}}{L_{1(n+1)}} & 0 & (A_1A_2)^2 & \cdots & (A_1A_{n+1})^2 \\ \frac{L_{2i}}{L_{2(n+1)}} & (A_2A_1)^2 & 0 & \cdots & (A_2A_{n+1})^2 \\ \cdots & \cdots & \cdots & \cdots & \\ \frac{L_{(n+1)i}}{L_{(n+1)(n+1)}} & (A_{n+1}A_1)^2 & \cdots & \cdots & 0 \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} 2 & \frac{2\Sigma L_{1i}^2}{L_{1(n+1)}^2} & \frac{2\Sigma L_{2i}^2}{L_{2(n+1)}^2} & \cdots & \frac{2\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}} \\ 1 & 0 & (A_1A_2)^2 & \cdots & (A_1A_{n+1})^2 \\ 1 & (A_2A_1)^2 & 0 & \cdots & (A_2A_{n+1})^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & (A_{n+1}A_1)^2 & \cdots & \cdots & 0 \end{vmatrix} \right]$$

developing these $n+1$ determinants of the $(n+2)^{th}$ order in products in pairs of the contituents of the first row and column it is easy to see that the simplification is only the common leading first minor.

Or $$(-1)^n 2^{n+1}(R.\, n!\, V)^2$$

$$= \begin{vmatrix} 0 & (A_1A_2)^2 & (A_1A_3)^2 & \dots & (A_1A_{n+1})^2 \\ (A_2A_1)^2 & 0 & (A_2A_3)^2 & \dots & (A_2A_{n+1})^2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (A_{n+1}A_1)^2 & (A_{n+1}A)^2 & (A_{n+1}A_3)^2 & \dots & 0 \end{vmatrix}$$

Also, from above, the coordinates of the centre, given by

$$\frac{-G_i}{F} = \frac{-G_i F}{F^2}$$

$$= \frac{(-1)^n}{2^{n+1}} \left[\frac{\Pi L_{i(n+1)}}{\Delta^n} \right]^2$$

$$\times \begin{vmatrix} 0 & 0 & \dots & 1 & \dots & 0 & 0 \\ \frac{\Sigma L_{1i}^2}{L_{1(n+1)}^2} & \frac{L_{11}}{L_{1(n+1)}} & \dots & \frac{L_{1i}}{L_{1(n+1)}} & \dots & \frac{L_{1n}}{L_{1(n+1)}} & 1 \\ \frac{\Sigma L_{2i}^2}{L_{2(n+1)}^2} & \frac{L_{21}}{L_{2(n+1)}} & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\Sigma L_{(n+1)i}^2}{L_{(n+1)(n+1)}^2} & \dots & \dots & \dots & \dots & \frac{L_{(n+1)n}}{L_{(n+1)(n+1)}} & 1 \end{vmatrix}$$

$$\times \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & \dfrac{-2\,L_{11}}{L_{1\,(n+1)}} & \dfrac{-2\,L_{12}}{L_{1\,(n+1)}} & \cdots & \dfrac{-2\,L_{1\,n}}{L_{1\,(n+1)}} & \dfrac{\Sigma L^2_{1i}}{L^2_{1\,(n+1)}} \\ 1 & \dfrac{-2\,L_{21}}{L_{2\,(n+1)}} & \dfrac{-2\,L_{22}}{L_{2\,(n+1)}} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \dfrac{-2\,L_{(n+1)\,1}}{L_{(n+1)\,(n+1)}} & \cdots & \cdots & \cdot & \dfrac{\Sigma L^2_{(n+1)i}}{L^2_{(n+1)\,(n+1)}} \end{vmatrix}$$

$$\frac{(-1)^{n+1}}{2^n (n!\,V)^2}$$

$$\times \begin{vmatrix} 0 & \dfrac{L_{1i}}{L_{1\,(n+1)}} & \dfrac{L_{2i}}{L_{2(n+1)}} & \cdots & \dfrac{L_{(n+1)i}}{L_{(n+1)\,(n+1)}} \\ 1 & 0 & (A_1A_2)^2 & \cdots & (A_1A_{n+1})^2 \\ 1 & (A_2A_1)^2 & 0 & \cdots & (A_2A_{n+1})^2 \\ \cdots & & \cdots & \cdots & \cdots \\ 1 & (A_{n+1}A_1)^2 & (A_{n+1}A_2)^2 & \cdots & 0 \end{vmatrix}^*$$

* *Cf.* Dr. Ganguli, '*Analytical Geometry of Hyperspaces*' Pt. 1, §§ 45, 18.

1

Bhāskarāchārya's References to Previous Teachers

BY

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In his article on the Source of Hindu Mathematics which appeared in the *Journal of the Royal Asiatic Society* for 1910, Mr. G. R. Kaye wrote as follows regarding Bhāskarāchārya:

"He even reproduces Brahmagupta's one example of 'fudging' and frequently in this section refers to 'ancient'* authorities and none of the cases so referred to can be traced to Hindu mathematicians'.' (p. 755).

In the same article he again writes: †

"Brahmagupta and Bhāskara distinctly indicate that they were compilers only, and frequent references are made by them ‡ to the 'text' and to 'ancient writers.' Colebrooke was misled into supposing that these ancient authorities were Hindus, but an examination of the

* Bhāskarāchārya uses the words पूर्वैः, आद्यैः, and पूर्वाचार्यैः which should be rendered as "by my predecessors," "by the first or original writers" and "by previous teachers." Mr. Kaye's rendering of these expressions as the 'ancients' leaves the matter indefinite and includes non-Hindu teachers, which is obviously against the spirit of the context. "Previous teachers" presumably refer to those of one's own country unless the contrary is suggested by the context.

† *J. R. A. S.*, p. 759.

‡ In the mathematical portions (*viz.*, Chapters XII, XVIII—XX) of Brahmagupta's work *Brāhma-sphuṭa-siddhānta* I have come across *only one* passage which seems to contain a reference to ancient teachers. It occurs in the obscure chapter on Permutations. It was only the Hindus who had been studying the subject from before the Christian era. Other peoples turned their attention to it several centuries after Bhāskarāchārya. The above reference must have, therefore, been to Hindu teachers. Will Mr. Kaye kindly quote the passages in which Brahmagupta is said to have made frequent references to the 'text' and to 'ancient teachers' as authorities for his mathematical rules?

references shows that the cases so referred to are just the cases that do not occur in earlier Hindu writings."

Again in 1911, Mr. Kaye wrote: * "Bhāskara often speaks with disdain of Hindu mathematicians and refers to certain 'ancient teachers' as authorities. If these ancient teachers had been Hindus, he would most probably have mentioned them by name."

These are some of the arguments which, in Mr. Kaye's opinion,† tend to prove the Greek or foreign influence on Hindu mathematics. The above arguments might be analysed thus:

(1) Bhāskara often speaks with disdain of Hindu mathematicians.

(2) The ancient authorities to whom Bhāskara refers were most probably foreigners as they have not been mentioned by name.

(3) None of the cases which Bhāskara refers to ancient teachers can be traced to Hindu mathematicians.

Here I have stated the arguments in their logical, and not in their chronological order. For, Mr. Kaye's articles are not likely to create a lasting impression separately. A reader of Mr. Kaye will consider all his arguments without any reference to the time when they were published.

Let us examine the above arguments in order.

(1) In the mathematical works of Bhāskarāchārya I have not come across even a single passage in which he may be said to have undoubtedly spoken of Hindu mathematicians with disdain. In his *Indian Mathematics* (1915, p. 21), Mr. Kaye writes:

'Bhāskara condemns them (*i.e.*, Brahmagupta, Mahāvīra, and Śrīdhara's rules relating to cyclic quadrilaterals, outright as unsound. "How can a person" he says "neither specifying one of the perpendiculars, nor either of the diagonals, ask the rest? Such a questioner is a blundering devil and still more so is he who answers the question."'

Perhaps the passage which has been translated by Mr. Kaye as above is the basis of his remark under consideration. This is apparently the only case of condemnation which he has been able to discover in the mathematical portions of Bhāskarāchārya's writings. But he does not hesitate in repeating (*East and West*, July, 1918, p. 678) his remark that Bhāskara 'often' speaks with disdain of Hindu mathematicians. On the other hand, there is not sufficient reason to suppose

* *Journal of the Asiatic Society of Bengal* (*J. A. S. B.*), 1911, p. 813.

† *East and West* (Simla), July, 1918, pp. 678 and 679.

that the passage referred to above contains any condemnation of Brahmagupta, Mahāvīra and Srīdhara. Bhāskara does not seem to have been aware of Mahāvīra's work. Towards the end of his *Vījagaṇita* Bhāskara acknowledges Brahmagupta, Srīdhara and Padmanābha's works as his sources. This fact has been noted also by Mr. Kaye in his *Indian Mathematics* (p. 37). He also writes that "Bhāskarāchārya's *Līlāvatī* is based on Srīdhara's work" (*Ind. Math.*, p. 24). Brahmagupta and Srīdhara make no mention of Āryabhaṭa's value of π but use an inaccurate value *viz.*, $\sqrt{10}$. If Bhāskara had a mind to speak with disdain of Brahmagupta and Srīdhara, he could not find a more suitable occasion than this. But what do we find? Bhāskara supports them by saying * that they took the value $\sqrt{10}$ for the sake of convenience (सुखार्थम्) and not because they did not know Āryabhaṭa's value (न हि ते न जानन्तीति). It is very difficult to reconcile Mr. Kaye's remark under consideration to these facts. Let us, therefore, examine the passage which seems to be the basis of Mr. Kaye's remark. It runs as follows : †

"लम्बयोः कर्णयोर्वैकमनिर्दिश्यापरान् कथम् ।
पृच्छत्यनियतत्वेऽपि नियतस्यापि तत्फलम् ॥
स पृच्छकः पिशाचो वा वक्ता वा नितरां ततः ।
यो न वेत्ति चतुर्बाहुक्षेत्रस्यानियतां स्थितिम् ॥"

Sudhākar Dvivedi's edition of the *Līlāvatī*, p. 44.

It may be rendered into English thus :

"Without specifying one of the perpendiculars or of the diagonals (of a quadrilateral with given sides), how can one ask to know the rest as well as the definite area inspite of the area being indefinite? The questioner who does not know that the area of a quadrilateral with given sides is indefinite is a devil and still more so is he who, being ignorant of the indefinite nature of the area of such a quadrilateral, answers the question."

It is difficult to believe that mathematicians like Brahmagupta and Srīdhara did not know that a quadrilateral with given sides cannot have a definite size and shape. Brahmagupta has given rules for finding the area and diagonals of a cyclic quadrilateral. But he has

* *Siddhānta-Śiromaṇi*, *Golādhyāya*, Chapter named *Bhubanakośa*, Bhāskara's own commentary on verse 52.

† Dr. Bibhutibhusan Datta has kindly informed me that here Bhāskara has echoed the remark made nearly two centuries earlier by Āryabhaṭa the Junior in the *Mahā-siddhānta* (Chap. XV, verse 70).

not expressly stated that the rules are applicable to cyclic quadrilaterals only. His rule for the area is

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}\,.$$

Bhāskarāchārya also has given this rule for the area of any quadrilateral; but he says that the area thus obtained cannot be exact. It would be well to quote his discussion of the point here. He writes :

"The diagonals of a quadrilateral with given sides are uncertain. How can, then, the area be definite? It is only in a set of quadrilaterals imagined (स्वकल्पिते*) by previous teachers and not in others that the diagonals found by them can exist. With the given sides the measures of the diagonals and the area can be manifold. For, if one pair of opposite vertices be drawn nearer, the diagonal joining them is shortened and the other diagonal lengthened, the vertices joined by it going asunder."

Is it unlikely that Brahmagupta and other mathematicians knew this fact? Even if we make the absurd supposition that the explanation given by Bhāskara for the uncertainty of the area of a quadrilateral with given sides might not strike Brahmagupta, we have got Āryabhaṭa to suggest it to him. Frequent references to Āryabhaṭa which occur in *Brāhma-sphuṭa-siddhānta* leave no room to doubt Brahmagupta's intimate acquaintance with Āryabhaṭa's work *Āryabhaṭiya*. The 13th verse † of

* In Sudhākar Dvivedi's edition of the *Līlāvatī* (Benares Sanskrit Series, No. 153) this word occurs as स्वकल्पितौ which I have changed into स्वकल्पिते for the following reasons :

(*a*) The word स्वकल्पितौ (imagined or assumed by one's self) can refer to the diagonals only. If the diagonals are first assumed or imagined, what is the necessity of finding them again? So, here the word cannot be used as such.

(*b*) The expression इतरत्र न स्तः means that 'they (*i.e.*, the diagonals found by previous teachers) do not exist elsewhere' So it must have been stated where they exist. Hence the necessity of the word स्वकल्पिते qualifying the noun चतुर्भुजे suggested by the context.

(*c*) This view is supported by a commentary written by Śrīdhara Mahāpātra more than 200 years ago. As found by the present writer in a manuscript, the commentary on the passage runs thus : तस्य चतुर्भुजस्य यौ श्रवणौ कर्णौ आद्यैः ब्रह्मगुप्तादिभिः नियतौ प्रसाधितौ तौ स्वकल्पितात् चतुर्भुजात् इतरत्र चतुर्भुजक्षेत्रे न स्तः भवतः। The form स्वकल्पिते seems to be better than स्वकल्पितात्. As इतरत्र has the suffix त्र denoting the seventh case, so the word स्वकल्पित must be in the seventh case.

† Mr. Kaye mistranslates त्रिभुज (a triangle) as a 'right-angled triangle,' चतुर्भुज (a quadrilateral) as a 'rectangle' and कर्ण (a diagonal) as a 'hypotenuse' (*J. A. S, B.*, March, 1908, p. 126).

its second chapter (*Gaṇitapāda*) states that a quadrilateral is determined by a diagonal (besides its sides). Similar remarks might apply to other Hindu mathematicians also. Hence the passage quoted above, far from condemning Brahmagupta and other Hindu mathematicians, does not even contain an indirect reflection on them. On the contrary, Bhāskarāchārya seems to support them against their detractors by suggesting that the rules were meant for a certain class of quadrilaterals only. This view is further supported by the fact that Bhāskarāchārya himself proposes simpler rules* than Brahmagupta and others for finding the diagonals of quadrilaterals which the latter had in view. What differentiates such quadrilaterals from others Bhāskarāchārya does not seem to know.

The passage quoted above is a general statement aimed, not against any particular person or persons but against those who might take the above mentioned rules of Brahmagupta to apply to any quadrilateral.

(2) If really Bhāskarāchārya had a low opinion of the Hindu mathematicians, it would be but natural to think of non-Hindu mathematicians when he referred to previous teachers as authorities without naming them. We have just now seen that there is no ground for thinking so. Mr. Kaye also knew it too well. He has, therefore, given another argument in support of his view. He says that, if the previous teachers had been Hindus, Bhāskarāchārya "would most probably have mentioned them by name." Bhāskarāchārya says† that the diagonals found by previous teachers exist only in quadrilaterals imagined by them. He does not name them at the place. But, for that very reason, we are not at all justified in holding that those teachers were non-Hindus. For, on a subsequent occasion (page 50) he mentions Brahmagupta as one of them. Just as it is absurd to think of outsiders when references are made in a public meeting to previous speakers so it is absurd to think of foreigners when an Indian mathematician refers to previous teachers.

(3) The third argument is the most effective and the boldness with which it has been urged is sure to mislead one who has not the opportunity to verify its correctness or cannot think of doubting the accuracy of a statement made by an investigator of the responsible position of Mr. Kaye. Colebrooke was a great Sanskrit scholar and had to teach Sanskrit in the Fort William College in Calcutta. He translated the mathematical works of the Hindus which were available in his time. When, therefore, Mr. Kaye writes that "Colebrooke was

* Sudhākar Dvivedi's edition of the *Līlāvatī* (1912), pp 51 and 52.

† *Ibid*, p. 44.

misled into supposing that these ancient teachers were Hindus," it is but natural for European and American scholars to regard Mr. Kaye as a great Sanskrit scholar with a greater degree of direct knowledge of the Hindu mathematical works. But this is far from true, as can be seen from Mr. Kaye's translation * of Āryabhaṭa's *Gaṇitapāda*. One should, therefore, be very cautious in accepting his interpretation of Sanskrit rules. His bold statement, *viz.*, "an examination of the references shows that the cases so referred to are just the cases that do not occur in earlier Hindu writings," in the face of Colebrooke's views to the contrary, cannot but betray a serious misreading of the fact as the following examination of Bhāskarāchārya's references will show.

In the *Līlāvatī* and the *Vījagaṇita* Bhāskarāchārya refers to unnamed previous teachers fourteen times only as noted below. Of the fourteen cases the first six occur in the *Līlāvatī* and the remaining eight in the *Vījagaṇita*.

[*N.B.*—Here the references are to Sudhākar Dvivedi's editions of the *Līlāvatī* (Benares, 1912, B. S. S. No. 153), the *Vījagaṇita* (Benares, 1888 , and *Brāhma-sphuṭa-siddhānta* (Benares, 1902).]

(*i*) In the verse† giving the names *eka*, *daśa*, *śata*, &c., up to *parārdha* Bhāskara states that these names of places have been coined by my predecessors (पूर्वैः). Were these names used similarly in any non-Hindu country before Bhāskarāchārya? Certainly, here by 'my predecessors', Bhāskara refers to Hindu authorities. Most of the above names have been used in the same sense by Āryabhaṭa, Mahāvīra and Śrīdhara.

(*ii*) In the rules ‡ giving the sum of the cubes of natural numbers beginning with unity Bhāskara says that it has been stated by the original writers (आद्यैः) that the sum in the question is equal to the square of the sum of those natural numbers. This rule has been given by Āryabhaṭa (vide *Āryabhaṭīya*, Chap. II, verse 22, and also *J. A. S. B*, 1908, p. 132) and Brahmagupta (vide *Brāhma-sphuṭa-siddhānta*, Section on Series, verse 20).

(*iii*) In the chapter on Mensuration (*Līlāvatī*, p. 44) Bhāskara says that the rules for finding the diagonals of a quadrilateral of known sides, as given by the previous teachers (आद्यैः), are applicable to particular cases of quadrilaterals and not to any quadrilateral. Such

* *J. A. S. B.*, March, 1908.

† *Līlāvatī*, p. 2.

‡ *Ibid*, p. 52.

rules occur in Brahmagupta's work *Brāhma-sphuṭa-siddhānta*, Section on Mensuration, verse 28). On page 50 of the *Līlāvatī* Bhāskara says distinctly that the rules were given by Brahmagupta and others.

(*iv*) On page 52 of the *Līlāvatī* Bhāskara again refers to the previous teachers (पूर्वैः) who gave the rules considered in (*iii*).

(*v*) The word प्रवदन्ति (*pravadanti*) which occurs in the rules for finding the chord of an arc of a circle and the diameter of the circle (*Līlāvatī*, p. 58) suggests that Bhāskara here refers to some previous teachers. The rule for the chord has been given by Āryabhaṭa (*Gaṇitapāda*, verse 17) and the other rule also follows easily from it. Both the rules occur in *Brāhma-sphuṭa-siddhānta* (Section on the circle, verse marked 41, *viz.*, the second verse in the section) as well as in Junior Āryabhaṭa's *Mahāsiddhānta* (chapter on Arithmetic, Rules 98 and 99).

(*vi*) Bhāskara concludes the chapter on shadow problems with a reference to the previous teachers. He says out of modesty that with the intention of increasing the intelligence of dull people like himself the learned have classified the different applications of the universal method of the Rule of Three as *prakīrṇa*, etc. Here Bhāskara evidently refers to Brahmagupta, Śrīdhara and others on whose works the *Līlāvatī* is based. As Śrīdhara's complete work is not extant, the *Līlāvatī* gives us some idea of the contents of its arithmetical portion.

(*vii*) The first reference to previous teachers (आचार्यवर्यैः) in Bhāskarāchārya's *Vījagaṇita* occurs in a passage (*Vījagaṇita*, page 8), which has been translated by Mr. Kaye as follows:—

"As many as (*yāvat tāvat*) and the colours 'black (*kālaka*), blue (*nīlaka*), yellow (*pītaka* and red (*lohitaka*)' and others besides these have been selected by ancient teachers for names of values of unknown quantities." (*Indian Mathematics*, p. 24.)

Mr. Kaye states in a foot-note that these teachers are not Indians. Although Brahmagupta does not explicitly state that colours or letters of the alphabet were chosen by him to represent unknown quantities, repeated occurrence of the word वर्ण (*varṇa*, which means both colours and letters of the alphabet in his rules * for the solution of equations involving two or more unknown quantities proves conclusively that he also represented unknown quantities by colours or letters of the alphabet. So, here Bhāskara certainly refers to Brahmagupta besides Śrīdhara

* *Brāhma-sphuṭa-siddhānta*. Sections on *Eka-varṇa-samīkaraṇa-vījam*, *Aneka-varṇa-samīkaraṇa-vījam*, and *Bhāvita-vījam*.

and Padmanābha as can be gathered from his own statement* as to the works on which his *Vijagaṇita* is based.

It is difficult to see how the Hindu use of *yāvat tāvat* (as many as) for an unknown quantity could be traced, as has been suggested † by Mr. Kaye to Diophantus' definition of the unknown quantity, *pléthos monáden aoriston*. Even if an illiterate boy, who has never heard of Diophantus, is asked "How many mangoes do you want to eat?", will it be unlikely for him to give the reply "*As many as* I can?" If the answer be in the negative as it certainly will, why should an Indian mathematician have to go to Diophantus for a similar answer? Again, Mr. Kaye makes‡ the absurd statement that the Hindu use of the names of colours for other unknown quantities was probably derived from the Chinese use of calculating pieces of two colours for positive and negative numbers, as if the use of colours were unknown in India. Mr. Kaye writes: § "the use of two such diverse types as *yāvat tāvat* and *kālaka* (generally abbreviated to *yā* and *kā*) in one system suggests the possibility of a mixed origin." Such use might betray lack of critical examination on the part of the Hindu mathematicians. But it does not prove their indebtedness to others. One who regularly reads reviews of books in periodicals often finds that even books written by eminent scholars of the present day show want of critical examination on the part of the authors. §§ We need not go far. Even Mr. Kaye who is very critical with respect to the mathematical and astronomical works of the Hindus is sometimes guilty of serious inconsistencies and self-contradictions. The writer had to refer to some elsewhere.** He now points out one more only which is a case in point. Mr. Kaye mis-translates †† Āryabhaṭa's verse for the modern arithmetical notation as follows:

"Units, tens, hunderds,............, thousands of millions. In these each succeeding place is ten times the preceding"

Here Mr. Kaye first names some numerical units and then, all on a sudden, discovers places and place-values in those units. "The use

* *Vijagaṇita*, p. 139 Also see Kaye, *Indian Mathematics*, p. 37.

† *Indian Mathematics*, p. 25.

‡ *Ibid.*

§ *Ibid*, pp 24 and 25

§§ For a recent case the reader is referred to the review of Dr. C. G. Abbot's *The Earth and the Stars*, which appears on page 819 of *Nature* (June 12, 1926) and specially to the last 15 lines of the right-hand column.

** *Journal of the Bihar and Orissa Research Society* for March, 1926, pp. 78-91.

†† *J. A. S. B.*, March, 1908, p. 117.

of two such diverse types as *yāvat tāvat* and *kālaka*" in one system is not more inexplicable than the curious mixture of numerical units and places in the above translation of a short verse. Mr. Kaye could not detect his own inconsistency in the above translation, which seems to have struck Fleet who has accordingly given * a consistent, though wrong interpretation of the verse. Are we, therefore, to suppose that the above translation is of a mixed origin and that the two parts were obtained by Mr. Kaye from two different translators?

The above inconsistency noted by Mr. Kaye has not escaped Bhāskarāchārya's notice. For, he writes † that, to avoid a mixture of two diverse types in one system, the letters beginninig with क (*ka*) —the initial consonant of the Devanagari alphabet—might be taken for the names of the unknown quantities (अथवा कादीन्यक्षराणि अव्यक्तानां संज्ञाः असंकरार्थं कल्प्याः). Here we have an important suggestion for the penultimate stage of what Nesselmann calls 'Symbolic Algebra,' as the letters suggested by Bhāskara can have no visible connection with the words or things which they are intended to represent.

The following explanation for the Hindu use of *yāvat tāvat* and the names of colours for unknown quantities is offered for what it is worth.

In equations involving one unknown quantity the Hindus appropriately used या (*yā*)—the initial letter of the expression *yāvat tāvat* (as many as)—for the unknown quantity. Such an equation has accordingly been called *eka-varṇa-samīkaraṇa* (an equation involving one letter). In the case of *aneka-varṇa-samīkaraṇa* (equations involving more than one letter) the Hindus had to find other letters besides या (*yā*). Hindu framers of verse have always been fond of the figure of speech known as यमक (*yamaka*) in which the same word is used in different senses in the same sentence. Unlike the English pun, it is used in serious writing. Hindu mathematicians were also expert in the composition of verse, as can be seen from their works. The word वर्ण (*varṇa*) means both colours and letters of the alphabet. So the Hindu mathematicians probably passed‡ from a letter to colours and

* *J. R. A. S.*, 1911, p. 116.

† *Vijagaṇita*, p. 112.

‡ Similarly phonetic resemblance led Āryabhaṭa to pass from *varga* places to *varga* consonants in devising his alphabetic system of expressing numbers. Similar is the idea underlying Brahmagupta's use of the word पद (*pada*) meaning a foot for the square *root*. The word पद (*pada*) when applied to a tree means its *root* (मूल). Also compare the use of the word श्रवण (*Śravaṇa*, *Līlāvatī* p. 44) meaning an ear for a diagonal of a quadrilateral. The Sanskrit word for a diagonal is कर्ण (*karṇa*) which also means an ear.

agreed to use the initial letters of the words *kālaka*, *nilaka*, *pītaka*, &c., for the other unknown quantities. As या (*yā*), का (*kā*), नी (*nī*), पी (*pī*) &c., are all *varṇa* whether in the sense of a letter or of a colour, the apparent inconsistency is not so serious as it appears to be at first sight. It is in the work of Āryabhaṭa that we first find equations involving more than one unknown quantity. Brahmagupta uses the terms *eka-varṇa-samīkaraṇa* and *aneka-varṇa--samīkaraṇa*. Therefore, either Āryabhaṭa or Brahmagupta used या (*yā*), का (*kā*), &c., for the unknown quantities. The later Hindu mathematicians only followed them by adopting this convention, as is also seen from a statement * of Bhāskarāchārya.

(*viii*) The next reference to previous teachers occurs in connection with the rules for finding the square and square root of irrational expressions (*Vījagaṇita*, p. 25). Here Bhāskara says that, as the intelligent can easily deduce these rules themselves, they have not been clearly explained by his predecessors (पूर्वैः). Āryabhaṭa has given a rule for the extraction of the square root of a rational number. It depends on the formula $(a+b)^2=a^2+2ab+b^2$. This formula has also to be used in finding the square of an irrational expression. Brahmagupta has given an imperfect treatment of irrationals in the section devoted to the addition of positive and negative quantities.

(*ix*) Bhāskarāchārya says (*Vījagaṇita*, pp. 73 and 92) that teachers (आचार्यैः) call an equation like

$$ax^2+b^2y^2+cz^2+dx+ey+fz=g$$

मध्यमाहरणम् (*madhyamāharaṇam*) and an equation like

$$axy+bx+cy=d$$

भावितम् (*Bhāvitam*).

Certainly these teachers cannot be non-Indians. Brahmagupta uses the word भावितम् in the same sense and has devoted a section to it. He also uses † the term मध्याहरण (*madhyaharana*) which means the same thing as मध्यमाहरण (*madhyamāharaṇa*). the latter term differing from the former in having one more suffix (*viz.*, म and one more prefix *viz.*, आ).

(*x*) Next Bhāskarācharya again acknowledges his indebtedness to previous teachers (पूर्वाचार्य :) for his notations for the unknown quantities (*Vija-gaṇita*, p. 111). *Vide* (*vii*) above.

* *Vījagaṇita*, p. 111. Also see (*x*) below.

† *Brahma-sphuṭa-Siddhanta*, Chapter on *Kuṭṭaka*, verse 2.

(*xi*) Then Bhāskarāchārya writes * that the knowledge (मतिः) which the previous teachers (आद्यैः) have extended with the help of various letters (विविधवर्णसंश्रयिणी) has now been given the name of *Vijagaṇita* (algebra). Certainly Brahmagupta, Śrīdhara, and Padmanābha are among those teachers. It was the Hindus who first used more than one letter to represent unknown quantities. Diophantus used a single letter † for his unknown quantities. Even in the thirteenth century A.D. the Chinese mathematicians did not use letters for unknown quantities but arranged their equations with red and black calculating pieces denoting the positive and negative co-efficients ; the unknown quantity had no symbol, the terms involving different powers of the unknown quantity being designated by the relative positions of their co-efficients. ‡ So here the reference cannot be to non-Hindu teachers.

(*xii*) Bhāskara quotes § an example given by some unknown person. From the language it appears that the author of the example must have been a Hindu mathematician who lived after Brahmagupta.

(*xiii*) Bhāskara next explains ‖ a rule given by previous teachers (पूर्वैः) for finding the integral solutions of equations of the form $y^2 = ax \pm b$. This very rule has not been given by Brahmagupta. But there is evidence ¶ to show that he has considered the case. The rule in question might have been given by Śrīdhara and Padmanābha. For, "Bhāskara at the end of his *Vijagaṇita* refers to the treatises on algebra of Brahmagupta, Śrīdhara and Padmanābha as 'too diffusive' and states that he has compressed the substance of them in 'a well reasoned compendium, for the gratification of learners'." ** The Greeks did not attempt to find integral roots of indeterminate equations †† except in a limited number of cases. Before the time of Brahmagupta the Chinese were acquainted with a method of solving indeterminate equations of the first degree only. Nearly a century after the composition of Brahmagupta's *Brahma-sphuṭa-siddhānta* (A.D. 628) the Chinese astronomer I-hsing contrived an arithmetical method called the *t'ai-yen shu* which, in the opinion of Mikami, "was a consideration

* *Vīja-gaṇita*, pp. 140 & 141.

† Heath, *History of Greek Mathematics*, Vol. II, p. 461.

‡ Mikami, *The Development of Mathematics in China and Japan*, p. 82.

§ *Vījagaṇita*, p. 154.

‖ *Ibid*, p. 168.

¶ *Brahma-sphuṭa-Siddhanta*, page 334, first example.

** *Indian Mathematics*, p. 37.

†† Heath, *History of Greek Mathematics*, Vol. II, p. 466.

in the indeterminate analysis." * But it is not known whether the Chinese dealt with indeterminate equations of the second degree before the time o Bhāskara.

(*iv*) The last reference to previous teachers (पूर्वाचार्यैः) occurs † in connection with the explanation of the rule for the solution in integers of indeterminate equations of the form $xy=ax+by+c$. This case as well as the more general case $mxy=ax+by+c$ has been considered by Brahmagupta in the section on *Bhāvita* (भावितबीजम्).

The object of this paper is not to disprove the foreign origin (if any) of Indian mathematics. The fact that Bhāskarāchārya referred to Indian mathematicians only cannot, by itself, establish indigenous origin. This paper, therefore, represents an attempt to correct some mis-statements which are being repeated from time to time and to give the reader some idea of the method which has been adopted by a critic of responsible position to make his preconceived anti-Indian views acceptable to his non-Indian readers.

* Mikami, *The Development of Mathematics in China and Japan*, p. 60.

† *Vija-gaṇita*, p. 176.

2

On the differentiability of a certain type of Integral Function

BY

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The object of the present paper is to consider the question of the differentiability of the integral function at a point where the integrand has a discontinuity of the second kind, the integrand being of the form $\frac{\sin}{\cos}\psi(x)$, where $\psi(x)$ is not *monotone* but makes infinite number of fluctuations between $+\infty$ and $-\infty$ as x tends to the point of discontinuity. It is believed that this question has not been considered in any previous publication.

For the sake of simplicity and fixity of ideas, I represent the integral function as

$$F(x)=\int_0^x f(t)dt$$

and take 0 to be the point of discontinuity at which the differentiability of $F(x)$ is to be considered. Also I restrict myself to the detailed consideration of only two types of integrands, *viz.*

$$\sin\frac{1}{\sin\frac{1}{t}} \quad \text{and} \quad \sin^2\left\{\log(\sin^2\log\tfrac{1}{t^2})\right\}.$$

In the first case $F'(0)$ exists and is zero; in the second case $F'(0)$ is proved to be non-existent. It is presumed that for the general case, in which

$$f(t)=\phi[\psi\{\phi\psi(t)\}],$$

ϕ being a periodic function with zero as one of its values, $F'(0)$ exists or not according as

$$\psi(t) \succ \log \frac{1}{t^2}$$

or not.

$$\textbf{Case}: f(t) = \sin \frac{1}{\sin \frac{1}{t}}$$

1. Suppose first that $x > 0$ and consider only those values of t which are positive. Then the zeros of

$$\sin \frac{1}{\sin \frac{1}{t}}$$

are given by

$$t = \frac{1}{R\pi \pm \sin^{-1} \frac{1}{S\pi}}$$

R and S being any positive integers.

If

$$\frac{1}{R_1\pi} \leq x < \frac{1}{(R_1 - 1)\pi},$$

where R_1 is a positive integer, then

$$F(x) \equiv \int_0^x f(t)dt = \int_{\frac{1}{R_1\pi}}^x f(t)dt + \sum_{R=R_1}^{\infty} \int_{\frac{1}{(R+1)\pi}}^{\frac{1}{R\pi}} f(t)dt \qquad \text{(A}$$

2. Consider

$$\int_{\frac{1}{(R+1)\pi}}^{\frac{1}{R\pi}} f(t)\, dt$$

It equals

$$\int_{\frac{1}{(R+1)\pi-\sin^{-1}\frac{1}{\pi}}}^{\frac{1}{R\pi+\sin^{-1}\frac{1}{\pi}}} f(t)dt + \sum_{S=1}^{\infty} \int_{\frac{1}{R\pi+\sin^{-1}\frac{1}{S\pi}}}^{\frac{1}{R\pi+\sin^{-1}\frac{1}{(S+1)\pi}}} f(t)dt$$

$$+ \sum_{S=1}^{\infty} \int_{\frac{1}{(R+1)\pi-\sin^{-1}\frac{1}{(S+1)\pi}}}^{\frac{1}{(R+1)\pi-\sin^{-1}\frac{1}{S\pi}}} f(t)\,dt$$

Now, making suitable transformations, the general terms of the first and second series are found to be equal to

$$(-1)^{R+S} \int_0^{\pi} \frac{\sin u\,du}{(S\pi+u)\sqrt{(S\pi+u)^2-1}\left(R\pi+\sin^{-1}\dfrac{1}{S\pi+u}\right)^2}$$

and

$$(-1)^{R+S} \int_0^{\pi} \frac{\sin u\,du}{(S\pi+u)\sqrt{(S\pi+u)^2-1}\left(R\pi-\sin^{-1}\dfrac{1}{S\pi+u}\right)^2}$$

respectively.

3. Thus

$$\sum_{R=R_1}^{\infty} \int_{\frac{1}{(R+1)\pi}}^{\frac{1}{R\pi}} f(t)\,dt = \sum_{R=R_1}^{\infty} \int_{\frac{1}{(R+1)\pi-\sin^{-1}\frac{1}{\pi}}}^{\frac{1}{R\pi+\sin^{-1}\frac{1}{\pi}}} f(t)\,dt$$

$$+(-1)^{R_1}\int_0^\pi \sin u\, du\, P(u)+(-1)^{R_1}\int_0^\pi \sin u\, du\, Q(u)$$

where

$$P=\sum_{\rho=0}^{\infty}(-1)^\rho\sum_{S=1}^{\infty}\frac{(-1)^S}{(S\pi+u)\sqrt{(S\pi+u)^2-1}\left\{(R_1+\rho)\pi+\sin^{-1}\dfrac{1}{S\pi+u}\right\}^2}$$

$$Q=\sum_{\rho=0}^{\infty}(-1)^\rho\sum_{S=1}^{\infty}\frac{(-1)^S}{(S\pi+u)\sqrt{(S\pi+u)^2-1}\left\{(R_1+\rho)\pi-\sin^{-1}\dfrac{1}{S\pi+u}\right\}^2}$$

4. Now denote each member of P by $(-1)^{\rho+S}\,T_{\rho,S}$; then it clear that the double series for P is absolutely, and, consequentl unconditionally convergent. Therefore the value of P is unaffected l taking the terms in the order

$$\begin{array}{l} -T_{0,1}+T_{0,2}-T_{0,3}+T_{0,4}-\ldots\ldots \\ +T_{1,1}-T_{1,2}+T_{1,3}-T_{1,4}+\ldots\ldots \\ -T_{2,1}+T_{2,2}-T_{2,3}+T_{2,4}\ldots\ldots \end{array}$$

etc.

But the terms in each column decrease numerically. Therefore

$$|P|<T_{0,1}+T_{0,2}+T_{0,3}+\ldots\ldots$$

$$<\frac{1}{R_1^2\pi^2}\sum_{S=S}^{\infty}\frac{1}{(S\pi+u)(\pi\overline{S-1}+u)}$$

$$i.e.\ \frac{1}{R_1^2\pi^3 u}.$$

Similarly it can be proved that

$$|Q|<\frac{1}{R_1^2\pi^3 u}.$$

Therefore

$$(-1)^{R_1}\int_0^{\pi}\sin u\,du\,(P+Q) \text{ is numerically less than } \frac{1}{R_1^{\,2}\pi}$$

where C_1 is a finite quantity.

5. Consider

$$\int_{\frac{1}{(R+1)\pi-\sin^{-1}\frac{1}{\pi}}}^{\frac{1}{R\pi+\sin^{-1}\frac{1}{\pi}}} \sin\frac{1}{\sin^{-1}\frac{1}{t}}\,dt.$$

It equals

$$\int_{\frac{1}{(R+1)\pi-\sin^{-1}\frac{1}{\pi}}}^{\frac{1}{R\pi+\frac{\pi}{2}}} \sin\frac{1}{\sin\frac{1}{t}}\,dt + \int_{\frac{1}{R\pi+\frac{\pi}{2}}}^{\frac{1}{R\pi+\sin^{-1}\frac{1}{\pi}}} \sin\frac{1}{\sin\frac{1}{t}}\,dt$$

$$=(-1)^{R+1}\int_1^{\pi} V_R\,du,$$

V_R standing for $\dfrac{2\sin u}{u\sqrt{u^2-1}\left(R\pi+\sin^{-1}\frac{1}{u}\right)^2}$.

Thus

$$\sum_{R=R_1}^{\infty}\int_{\frac{1}{(R+1)\pi-\sin^{-1}\frac{1}{\pi}}}^{\frac{1}{R\pi+\sin^{-1}\frac{1}{\pi}}} f(t)\,dt=(-1)^{R_1+1}\int_1^{\pi} W\,du$$

$$W=2\sum_{s=0}^{\infty}\frac{(-1)^{s}\sin u}{u\sqrt{u^2-1}\left\{(R_1+s)\pi+\sin^{-1}\frac{1}{u}\right\}^2},$$

which is numerically less than

$$\frac{2\sin u}{u\sqrt{u^2-1}}\cdot\frac{1}{R_1^{\,2}\pi^2}.$$

Therefore the infinite series in (1) is numerically less than

$$\frac{C_2}{R_1^{\,2}\pi^3}$$

where C_2 is a finite constant.

Similarly it can be proved that

$$\int_{\frac{1}{R_1\pi}}^{\infty} f(t)\,dt$$

is numerically less than

$$\frac{C_3}{(R_1-1)^2\pi^3},$$

C_3 being a finite constant.

6. From the results given above about each member of the right hand expression of equation (A), it follows that

$$F(x)=\frac{C}{(R_1-1)^2\pi^2},$$

...finite constant.

$$x\leqq\frac{1}{R_1\pi}.$$

$$\frac{F(x)-F(0)}{x}=\frac{F(x)}{x}$$

numerically less than

$$\frac{|C|.\,R_1}{(R_1-1)^2\pi}.$$

Therefore

$$\lim_{x=+0}\frac{F(x)}{x}=0.$$

Similarly the case when x is negative can be considered and $F'(-0)$ can be proved to be existent and zero.

Therefore it is proved that $F'(0)$ exists and is zero.

$$\textbf{Case}: \quad f(t)=\sin^2\left\{\log\left(\sin^2\log\frac{1}{t^2}\right)\right\}.$$

7. Suppose first that x is positive and therefore consider only positive values of t. The zeros of the integrand are given by

$$t=e^{-\frac{1}{2}\left(R\pi\pm\sin^{-1}e^{-\frac{S\pi}{2}}\right)},$$

R and S being any positive integers, including zero.

Further, let

$$e^{-\frac{1}{2}R_1\pi}\leq x<e^{-\frac{1}{2}(R_1-1)\pi},$$

where R_1 is a positive integer.

Then

$$F(x)=\sum_{R=R_1}^{\infty}\int_{e^{-\frac{1}{2}(R+1)\pi}}^{e^{-\frac{1}{2}R\pi}}f(t)dt+\int_{e^{-\frac{1}{2}R_1\pi}}^{x}f(t)dt.$$

8. Consider

$$\int_{e^{-\frac{1}{2}(R+1)\pi}}^{e^{-\frac{1}{2}R\pi}}f(t)dt=\sum_{S=0}^{\infty}\int_{e^{-\frac{1}{2}\left\{(R+1)\pi-\sin^{-1}e^{-(S+1)\frac{\pi}{2}}\right\}}}^{e^{-\frac{1}{2}\left\{(R+1)\pi-\sin^{-1}e^{-\frac{S\pi}{2}}\right\}}}f(t)dt$$

$$+\sum_{S=0}^{\infty}\int_{e^{-\frac{1}{2}\left\{R\pi+\sin^{-1}e^{-\frac{S\pi}{2}}\right\}}}^{e^{-\frac{1}{2}\left\{R\pi+\sin^{-1}e^{-(S+1)\frac{\pi}{2}}\right\}}}f(t)dt$$

Denoting the first and second parts of the right hand expression n the above equation by I_R and J_R respectively, we have, by naking suitable transformations, the general term in the series I_R as

$$\frac{1}{4}\int_0^{\pi}\frac{\sin^2 u du}{\sqrt{1-e^{-(S\pi+u)}}}\cdot e^{-\frac{1}{2}\left\{(R+1)\pi+(S\pi+u)+\sin^{-1}\left(e^{-\frac{S\pi+u}{2}}\right)\right\}}$$

ınd the general term in the series J_R is similarly found to be

$$\frac{1}{4}\int_0^{\pi}\frac{\sin^2 u du}{\sqrt{1-e^{-(S\pi+u)}}}\cdot e^{-\frac{1}{2}\left\{R\pi+(S\pi+u)+\sin^{-1}\left(e^{-\frac{S\pi+u}{2}}\right)\right\}}.$$

Thus

$$I_R + J_R = \frac{1}{4}e^{-\frac{1}{2}R\pi} \times C,$$

where C is a finite constant equal to

$$\int_0^{\pi}\sin^2 u.\ P(u)du,$$

$P(u)$ standing for

$$\sum_{S=0}^{\infty}\frac{e^{-\frac{1}{2}\left\{(1+S)\pi+u-\sin^{-1}\left(e^{-\frac{S\pi+u}{2}}\right)\right\}}+e^{-\frac{1}{2}\left\{(S\pi+u)+\sin^{-1}\left(e^{-\frac{S\pi+u}{2}}\right)\right\}}}{\sqrt{1-e^{-(S\pi+u)}}}$$

9. Let $$x=e^{-\frac{1}{2}R_1\pi},$$
then

$$F(x)=C\times\frac{1}{4}\sum_{R=R_1}^{\infty} e^{-\frac{1}{2}R\pi}$$

$$=C\times\frac{1}{4}\,\frac{e^{-\frac{1}{2}R_1\pi}}{1-e^{-\frac{1}{2}\pi}}.$$

Therefore, if x tends to zero (with increasing R_1), the limit of

$$\frac{F(x)-F(0)}{x}\equiv\frac{F(x)}{x}$$

is
$$\frac{C}{4(1-e^{-\frac{1}{2}\pi})}.$$

10. Next, let

$$x=e^{-\frac{1}{2}R_1\pi+\frac{1}{2}\sin^{-1}\left(e^{-\frac{\pi}{2}}\right)}.$$

Then it is easily seen that

$$\int_{e^{-\frac{1}{2}R_1\pi}}^{x} f(t)dt=+\frac{1}{4}e^{-\frac{1}{2}R_1\pi}\times C_1,$$

where C_1 is equal to

$$\int_0^{\pi} du\,\sin^2 u\sum_{S=0}^{\infty}\frac{e^{-\frac{1}{2}\left[(1+S)\pi+U\,\sin^{-1}\left(e^{-\frac{(1+S)\pi+u}{2}}\right)\right]}}{\sqrt{1-e^{-[(1+S)\pi+u]}}}$$

Therefore

$$F(x)=\left\{\frac{C}{4\left(1-e^{-\frac{\pi}{2}}\right)}+\frac{C_1}{4}\right\}e^{-\frac{1}{2}R_1\pi}$$

Hence

$$\frac{F(x)-F(0)}{x}=\frac{F(x)}{x}=\left\{\frac{C}{4\left(1-e^{-\frac{\pi}{2}}\right)}+\frac{C_1}{4}\right\}e^{-\frac{1}{2}\sin^{-1}e^{-\frac{\pi}{2}}}$$

Therefore, if x tends to zero (with increasing R_1) the limit of

$$\frac{F(x)-F(0)}{x}$$

is certainly different from that in the case considered in the preceding paragraph.

Therefore $F'(+0)$ does not exist.

Similarly it can be proved that $F'(-0)$ does not exist.

us it is proved that

$$\int_0^x \sin^2 \left\{ \log \left(\sin^2 \log \frac{1}{t^2} \right) \right\} dt$$

differential co-efficient at $x=0$.

Conclusion

The results given in the preceding paragraphs easily admit of lization in a number of ways.

If there are a *finite* number of steps, say n, in the expression

$$f(t) \equiv \sin \frac{1}{\sin \frac{1}{\sin \frac{1}{\cdots}}} \quad \cdots \quad \sin \frac{1}{t}$$

$F'(0)$ exists and is 0, whatever n may be.

Similarly, for the case

$$f(t)=\sin^2 \log \sin^2 \log \sin^2 \log \ldots . . \sin^2 \log \frac{1}{t^2},$$

s non-existent.

The procedure adopted in the preceding paragraphs will be ole to the general case in which

$$f(t)=\phi[\psi\{\phi\psi(t)\}],$$

any periodic function with zero as one of its values and $\psi(t)$ onotone with the condition that

$$\psi(t) \succ 1.$$

presumed that $F'(0)$ is existent or not according as

$$\psi(t) \succ \log \frac{1}{t^2}$$

, Cal. Math. Soc., Vol. XVIII, No. 2.

On sound waves due to prescribed vibrations of a cylindrical surface in the presence of a rigid and fixed cylindrical obstacle

BY

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The problem of the scattering of sound waves by an obstacle when the source producing the waves and the nature of the obstacle are given, has been solved for only a limited number of cases although the problem attracted the attention of mathematicians as early as 1862. Among the distinguished mathematicians who have considered this problem may be mentioned the names of Clebsch*, Rayleigh† and Stokes‡. Recently S. K. Banerjee§ has given a solution of the problem for the case in which the source is a spherical surface of which the vibrations are prescribed and the obstacle is a rigid and fixed spherical surface. The analogous problem in which the source and the obstacle are both infinitely long circular cylinders does not appear to have been investigated by any previous author. In the present paper it is proposed to give a solution of this problem.

2. Let a be the radius of the vibrating cylinder, b that of the fixed one and let d denote the distance between their axes which are supposed to be parallel.

* Clebsch : "Über die Reflexion an einer kugelfläche," *Jour. f. Math.*, Bd. 61 p. 68 (1863).

† Rayleigh : "Investigation of the disturbances produced by a spherical obstacle on the waves of Sound," *Proc. Lond. Math. Soc.*, Vol. 4 (1872).

‡ Stokes : *Mathematical and Physical Papers*, Vol. 4, p. 314.

§ S. K. Banerjee : "On Sound Waves etc. etc." *Bull. Cal. Math. Soc.*, Vol. 4, (1912-13).

The cylinders being supposed to be infinitely long and the prescribed vibrations being taken to be transverse to the axis, the problem under consideration will be a two-dimensional one.

The normal component of the prescribed vibration at any point on the cylinder can obviously be expanded in a Fourier Series, so that, if $\frac{2\pi}{\sigma}$ be the period, we can assume for the normal velocity an expression of the form

$$\sum_{n=0}^{\infty} (u_n \cos n\theta + v_n \sin n\theta)\, e^{i\sigma t}$$

Let C and C′ be the centres of the two circles in any normal section and let (r,θ) and (r',θ') denote the cylindrical polar co-ordinates of any point referred to C and C′ respectively as origins and CC′ and C′C respectively as initial lines.

If ϕ denote the velocity potential of the sound waves and c the velocity of propagation appropriate to the surrounding medium which is taken to be frictionless, ϕ will satisfy the following equations :

$$\nabla_1^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \qquad \ldots \quad (1)$$

at all points of the surrounding medium, ∇_1^2 standing for either of the operators

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\text{or} \quad \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2}{\partial \theta'^2};$$

$$\frac{\partial \phi}{\partial r} = -\sum_{0}^{\infty}(u_n \cos n\theta + v_n \sin n\theta)\, e^{i\sigma t}, \text{ when } r=a; \quad \ldots \quad (2)$$

$$\text{and } \frac{\partial \phi}{\partial r'} = 0, \text{ when } r' = b. \qquad \ldots \quad (3)$$

3. The following transformation theorems* will be required fo establishing the results obtained below :

Theorem I. If J_n denotes a Bessel Function of the *first kind and o integral order n*, then whether r' is *greater or less* than d,

$$J_n(r) \frac{\cos}{\sin} n\theta = \sum_{m=-\infty}^{\infty} J_{n+m}(d) J_m(r') \frac{\cos}{\sin} m\theta'.$$

Theorem II. If C_n denotes *any* cylinder function, then *if* $r' < d$,

$$C_n(r) \frac{\cos}{\sin} n\theta = \sum_{m=-\infty}^{\infty} C_{n+m}(d) J_m(r') \frac{\cos}{\sin} m\theta'$$

Theorem III. If C_n denotes any cylinder function of *integral order n other than a function of the first kind*, then *if* $r' > d$,

$$(-)^n C_n(r) \cos n\theta = \sum_{m=-\infty}^{\infty} J_m(d) C_{n+m}(r') \cos (n+m)\theta'$$

and $$(-)^n C_n(r) \sin n\theta = \sum_{m=-\infty}^{\infty} J_m(d) C_{n+m}(r') \sin (n+m)\theta'.$$

The above theorems are true if for r, r' and d we write kr, kr' and kd, where k is any constant factor. They also hold good if (r, θ) and (r', θ') are interchanged and the restrictions imposed on r' are replaced by similar restrictions on r.

4. Assuming the motion of the gas to be regularly periodic, we can now write

$$\phi = \psi \, e^{ikct}, \qquad \text{where} \quad kc = \sigma,$$

ψ being a function of r and θ only.

* For Theorems I and II see Watson's *Treatise on Bessel Functions* (1922), p. 361, where the results have been obtained from Graf's generalised addition formula. Theorem III can be easily established from Graf's formula.

Then the equation (1) reduces to

$$(\nabla_1^2+k^2)\psi=0. \qquad \dots \quad (4)$$

Supposing ψ to be developed in a Fourier Series, it appears from the equations (2) and (4) that ψ will be of the form

$$\sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)\psi_n,$$

where ψ_n is the general solution of the equation

$$\frac{\partial^2\psi_n}{\partial r^2}+\frac{1}{r}\,\frac{\partial\psi_n}{\partial r}+\left(k^2-\frac{n^2}{r^2}\right)\psi_n=0, \qquad \dots \quad (5)$$

which is Bessel's equation of order n.

Also the equations (2) and (3) require that

$$\frac{\partial\psi}{\partial r}=-\sum_{n=0}^{\infty} (u_n \cos n\theta + v_n \sin n\theta) \quad \text{when} \quad r=a, \qquad \dots \quad (6)$$

and

$$\frac{\partial\psi}{\partial r'}=0, \qquad \text{when} \quad r'=b. \qquad \dots \quad (7)$$

5. Consider at first the case in which the cylinders are external to each other. It is obvious that the solution must represent a system of diverging waves. Hence the appropriate solution * of the equation (5) is, for this case,

$$\psi_n=H_n^{(2)}(kr).$$

* See Lamb's *Hydrodynamics* (5th edition) p. 274 and also p. 502. The function $H_n^{(2)}$ is the second of the two functions called Hankel's Functions of the Third Kind by Watson. (See Watson's *Treatise on Bessel Functions*, p. 73). It can be easily verified that $H_n^{(2)}=iD_n$, where D_n is the function used by Lamb for representing diverging waves. Rayleigh also has used a notation similar to Lamb's. See *Scientific Papers*, Vol. 4, p. 290; also Vol. 5, pp. 410-418.

For the sake of convenience we shall, in what follows, write H_n for $H_n^{(2)}$.

Thus the function

$$\psi_0 = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) H_n(kr)$$

would be the proper value for ψ if it satisfies the equations (6) and (7).

The equation (6) would be satisfied if we took

$$A_n = -\frac{u_n}{kH_n'(ka)}$$

and
$$B_n = -\frac{v_n}{kH_n'(ka)}.$$

We take these values for A_n and B_n. But $\psi=\psi_0$ does not satisfy the equation (7).

To satisfy this equation, assume

$$\psi = \psi_0 + \psi_1,$$

where ψ_1 is such that in addition to satisfying the equation (4) it makes

$$\frac{\partial \psi_0}{\partial r'} + \frac{\partial \psi_1}{\partial r'} = 0, \qquad \text{when} \quad r' = b. \qquad \dots \quad (8)$$

Since ψ_1 would in reality represent the waves scattered from the cylinder $r'=b$ after the incidence of the system represented by ψ_0, ψ_1 must also represent a system of diverging waves and we may assume

$$\psi_1 = \sum_{m=-\infty}^{\infty} (A_m^{(1)} \cos m\theta' + B_m^{(1)} \sin m\theta') H_m(kr')$$

neighbourhood of the cylinder of radius b, $r' < d$,
›rem II, in this neighbourhood

$$\sum_{=-\infty}^{\infty} J_m(kr')\Big[\cos m\theta' \sum_{n=0}^{\infty} A_n H_{n+m}(kd)$$

$$+ \sin m\theta' \sum_{n=0}^{\infty} B_n H_{n+m}(kd)\Big].$$

ondition (8), and therefore the condition (7), will be

$$= -\frac{J'_m(kb)}{H'_m(kb)} \sum_{n=0}^{\infty} A_n H_{n+m}(kd)$$

$$= \frac{J'_m(kb)}{kH'_m(kb)} \sum_{n=0}^{\infty} \frac{u_n H_{n+m}(kd)}{H'_n(ka)},$$

$$= -\frac{J'_m(kb)}{H'_m(kb)} \sum_{n=0}^{\infty} B_n H_{n+m}(kd)$$

$$= \frac{J'_m(kb)}{kH_m'(kb)} \sum_{n=0}^{\infty} \frac{v_n H_{n+m}(kd)}{H_n'(ka)}.$$

$_1$ would no longer satisfy the condition (6). We then
:on ψ_2 such that it represents a system of diverging
ng in fact the system scattered by the cylinder $r=a$
:e of the system ψ_1, and also satisfying the equation

$$\frac{\partial \psi_2}{\partial r} = 0, \quad \text{when} \quad r=a. \quad \dots \quad \dots \quad (9)$$

neighbourhood of the cylinder $r=a$,

$$J_p(kr)\Big[\cos p\theta \sum_{m=-\infty}^{\infty} A_m^{(1)} H_{m+p}(kd)$$

$$+ \sin p\theta \sum_{m=-\infty}^{\infty} B_m^{(1)} H_{m+p}(kd)\Big]$$

Therefore assuming

$$\psi_2 = \sum_{n=-\infty}^{\infty} \mathrm{H}_p(kr) \left[\mathrm{A}_p^{(2)} \cos p\theta + \mathrm{B}_p^{(2)} \sin p\theta \right].$$

we have in order that the equation (9) may be satisfied

$$\mathrm{A}_p^{(2)} = -\frac{\mathrm{J}_p'(ka)}{\mathrm{H}_p'(ka)} \cdot \sum_{n=-\infty}^{\infty} \mathrm{A}_n^{(1)} \mathrm{H}_{n+p}(kd)$$

and

$$\mathrm{B}_p^{(2)} = -\frac{\mathrm{J}_p'(ka)}{\mathrm{H}'_p(ka)} \cdot \sum_{n=-\infty}^{\infty} \mathrm{B}_n^{(1)} \mathrm{H}_{n+p}(kd)$$

Proceeding in this way we find

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots\ldots\ldots\ldots\ldots\ldots,$$

where ψ_0 represents the original waves diverging from the cylinder $r=a$ and ψ_1, ψ_2 etc., represent the waves due to successive reflections from the two cylinders.

Thus finally we have

$$\phi = (\psi_0 + \psi_1 + \psi_2 + \ldots\ldots\ldots\ldots\ldots\ldots)e^{i\sigma t}$$

as the velocity potential of the entire system of waves generated.

6. Consider now the case in which the vibrating cylinder $r=a$ is within the cylinder $r'=b$. In this case if the waves originally diverging from the cylinder $r=a$ be represented by the velocity potential function $\psi_0 e^{i\sigma t}$ we have, as before,

$$\psi_0 = \sum_{n=0}^{\infty} (\mathrm{A}_n \cos n\theta + \mathrm{B}_n \sin n\theta) \mathrm{H}_n(kr),$$

where $k = \frac{\sigma}{c}$, $\mathrm{A}_n = -\frac{u_n}{k\mathrm{H}'_n(ka)}$,

and $\mathrm{B}_n = -\frac{v_n}{k\mathrm{H}'_n(ka)}$

If ψ_1 represent the function corresponding to the waves reflected for the first time from the cylinder $r'=b$ then having regard to the fact that the space to which this function relates is that internal to the cylinder $r'=b$, we may assume

$$\psi_1 = \sum_{p=-\infty}^{\infty} \left[A_p^{(1)} \cos p\theta' + B_p^{(1)} \sin p\theta' \right] J_p(kr').$$

The coefficients are to be determined from the condition that, when $r'=b$, we must have

$$\frac{\partial \psi_0}{\partial r'} + \frac{\partial \psi_1}{\partial r'} = 0 \qquad \ldots \qquad \ldots \qquad \ldots \quad (10)$$

Since $r'>d$ in the neighbourhood of $r'=b$, we have, by Theorem III of Art. 8, in this neighbourhood

$$\psi_0 = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} (-)^n J_m(kd) \left\{ A_n \cos(n+m)\theta' - B_n \sin(n+m)\theta' \right\}$$

$$\times\ H_{n+m}(kr')$$

$$= \sum_{p=-\infty}^{\infty} (-)^p H_p(kr') \left[\cos p\,\theta' \sum_{n=0}^{\infty} A_n J_{n-p}(kd) \right.$$

$$\left. - \sin p\theta' \sum_{n=0}^{\infty} B_n J_{n-p}(kd) \right].$$

Hence from (10) we have

$$A_p^{(1)} = -(-)^p \cdot \frac{H'_p(kb)}{J'_p(kb)} \cdot \sum_{n=0}^{\infty} A_n J_{n-p}(kd)$$

and

$$B_p^{(1)} = (-)^p \cdot \frac{H'_p(kb)}{J'_p(kb)} \cdot \sum_{n=0}^{\infty} B_n J_{n-p}(kd)$$

Now the waves represented by the function ψ_1 will, after incidence on the cylinder $r=a$, be reflected, and if ψ_2 denote the function corresponding to these reflected waves, we may assume

$$\psi_2 = \sum_{s=-\infty}^{\infty} \left[A_s^{(2)} \cos s\theta + B_s^{(2)} \sin s\theta \right] H_s(kr)$$

The function ψ_2 must satisfy the equation

$$\frac{\partial \psi_1}{\partial r} + \frac{\partial \psi_2}{\partial r} = 0, \quad \text{when} \quad r=a. \quad \ldots \qquad \ldots \qquad (11$$

Now the value of ψ_1 in the neighbourhood of the cylinder $r=a$, where r may be either greater or less than d, may be easily expressed in terms of r, θ by means of Theorem I of Art. 3. Thus in the neighbourhood of $r=a$,

$$\psi_1 = \sum_{s=-\infty}^{\infty} J_s(kr) \Big[\cos s\,\theta \sum_{p=-\infty}^{\infty} A_p^{(1)} J_{p+s}(kd)$$

$$+ \sin s\theta \sum_{p=-\infty}^{\infty} B_p^{(1)} J_{p+s}(kd) \Big]$$

Substituting this value of ψ_1 in the equation (11), the coefficients $A_s^{(2)}$, $B_s^{(2)}$ can be easily obtained.

Obtaining in this way the functions ψ_3, ψ_4, etc., due to the successively reflected waves, we have, for the velocity potential function of of the entire system of waves generated,

$$\phi = (\psi_0 + \psi_1 + \psi_2 + \ldots\ldots\ldots\ldots\ldots\ldots) e^{i\sigma t}$$

7. If the cylinder $r=a$ be internal to the cylinder $r=b$ and at the same time co-axial with it, then the boundary conditions become

$$\frac{\partial \psi}{\partial r} = -\sum_{0}^{\infty} (u_n \cos n\theta + v_n \sin n\theta), \text{ when } r=a$$

A NOTE ON HURWITZ'S PAPER ON THE EXPANSION COEFFICIENTS OF A PARTICULAR LEMNISCATE FUNCTION

BY

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The late Professor A. Hurwitz* has given the expansion of $\wp(u)$ in powers of u in the form

$$\wp(u)=\frac{1}{u^2}+\frac{2^4E_1}{4}\cdot\frac{u^2}{2!}+\frac{2^8E_2}{8}\cdot\frac{u^6}{6!}+\ldots\ldots+\frac{2^{4n}E_n}{4n}\frac{u^{4n-2}}{(4n-2)!}+\ldots\ldots$$

where $\wp(u)$ satisfies the differential equation

$$\wp'^2=4\wp^3-4\wp.$$

The numbers E_1, E_2... etc., are analogous to Bernoulli's numbers.

From $\wp''=6\wp^2-2$, we have $E_1=\frac{1}{10}$

and for $n>1$

$$E_n=\frac{3}{(2n-3)(16n^2-1)}\sum_{k=1}^{n-1}(4k-1)(4n-4k-1)\,{}^{4n}C_{4k}\,E_kE_{n-k}.$$

The denominator of E_n contains only single prime factors, and indeed besides the prime 2 all and only those primes p of the form $4k+1$ for which $p-1$ is a divisor of $4n$.

Further

$$E_n=G_n+\frac{1}{2}+\sum\frac{(2a)^{\frac{4n}{p-1}}}{p} \qquad \ldots \quad (1)$$

where the summation extends over the above mentioned primes, whilst G_n denotes an odd integral number.

In the present paper, I have verified that the formula (1) holds good as far as E_{11}, thus being in agreement with the results† I obtained for the expansion of $\wp(u)$ in powers of u.

* *Göttingen Nachrichten*, (1897), pp. 273-275.

† *Bull. Cal. Math. Soc*, Vol. XVII, No. 4.

$$E_1=\frac{1}{10}=\frac{1}{2}-\frac{2}{5}.$$

$$E_2=\frac{3}{10}=-1+\frac{1}{2}+\frac{4}{5}.$$

$$E_3=\frac{3^4.7}{130}=5+\frac{1}{2}-\frac{8}{5}+\frac{6}{13}.$$

$$E_4=\frac{3^4.7^2.11}{170}=253+\frac{1}{2}+\frac{16}{5}+\frac{2}{17}.$$

$$E_5=\frac{3^6.7^2.11}{10}=39299+\frac{1}{2}-\frac{32}{5}.$$

$$E_6=\frac{3^7.7^3.11^2.19}{130}=13265939+\frac{1}{2}+\frac{64}{5}+\frac{36}{13}.$$

$$E_7=\frac{3^9.7^4.11^4.19.23}{290}=8616924013+\frac{1}{2}-\frac{128}{5}-\frac{10}{29}.$$

$$E_8=\frac{3^{10}.7^4.11.19.23.2453}{170}=9833937781275+\frac{1}{2}+\frac{256}{5}+\frac{4}{17}.$$

$$E_9=\frac{3^{14}.7^4.11^3.19.23.31.427}{4810}$$

$$=18382040180023477+\frac{1}{2}-\frac{512}{5}+\frac{216}{13}-\frac{12}{37}.$$

$$E_{10}=\frac{3^{13}.7^5.11^3.19^2.23.31\ 2381}{410}$$

$$=53311001020080183933+\frac{1}{2}+\frac{1024}{5}+\frac{10}{41}.$$

$$E_{11}=\frac{3^{15}.7^6.11^4.19.23.31.6859}{10}$$

$$=229658082900486063068939+\frac{1}{2}-\frac{2048}{5}.$$

My best thanks are due to Dr. G. Prasad, D.Sc., who kindly suggested the investigation to me.

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Triadic Equations in Hyperbolic Geometry

BY

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1. Introduction.

The present paper is an application and development of the principles explained and developed in the paper "*General Theorem of Symmetrics*" published in the *Bulletin of the Calcutta Mathematical Society*, Vol. XXVI, No. 1, 1926 and should be read for a proper understanding along with that paper.* A short resume however is given of the principles explained and the notations used in that paper so that in a manner it can be followed independently of that paper. The '*triadic coordinates*' introduced and so named in this paper differ from '*Wierstrassian coordinates*' mainly in the fact that any of the elements whose coordinates are united in any equation may be indifferently a point or a line. Thus points and lines stand in a relation of unity and not of duality.

2. Definitions.

We will denote the *point*, the *line* and the *horocycle* as *basic elements*.

All horocycles having the same system of axes will be considered equivalent as representing the same basic element, which is a conceptual point at infinity to which all the axes converge.

A point or a line as a basic element can have *associated* with it two *directed elements* of opposite senses. The horocycle as a basic element stands quite *isolated* in this respect.

* The use of oriented points and lines which was first made in the paper above referred to and has been maintained in this, forms a special feature of this paper. T. Takasu of the Tohoku Imperial University in an elegant paper "Natural Non-Euclidean Geometry, Doubly Oriented Points, Lines and Planes as Elements" published in the Tohoku Mathematical Journal of April, 1926, has developed the theory of orientation of points, lines and planes in Non-Euclidean 3-space.

To a basic line element can be associated two *directed line elements* having the same position but opposite senses, *i.e.*, directions of imaginary translation along them.

To a basic point element can be associated two *directed point elements* having the same position but opposite senses, *i.e.*, directions of imaginary rotation about them.

The two directed elements associated with a basic point or a basic line will be called its *orients*. Of these if one be called the *positive orient* the other will be called the *negative orient*.

If a be a basic element, a point or a line, its two orients will be denoted by a_+ and a_- and either of them by a_0.

Two basic lines will be called *intimate* if they are at right angles. A basic point and a basic line will be called *intimate* if the latter passes through the former. A basic line is *intimate* with a horocycle if the former is an axis of the latter. A horocycle will be called *intimate* with itself or any equivalent horocycle. It may be observed that two basic points cannot be intimate, neither can a basic point and a horocycle be intimate under any circumstances.

The *join** of two basic elements is a third basic element intimate with both. It will be observed that a unique join exists in every case. If a and β be any two basic elements then $(a\,\beta)$ will represent their join. Similarly the join of γ with $(a\,\beta)$ will be represented by $\{(a\beta)\gamma\}$ and the join of $(a\beta)$ with $(\gamma\delta)$ by $\{(a\beta)'\gamma\delta)\}$ and so on.

Any three elements will be called *co-intimate* if there is a common element intimate with each.

The sense of a directed line relative to a point not lying on it may be clockwise or counterclockwise. Similarly the sense of a directed point about the point itself may be clockwise or counterclockwise.

If the senses of two directed points are both clockwise or both counterclockwise they are said to be *similarly oriented*, but if one of the senses be clockwise and the other counterclockwise they are said to be *oppositely oriented.*

If the senses of a directed line and a directed point be such that the sense of the former relative to the base of the latter and the sense of the latter itself are both clockwise or both counter-clockwise they are said to be *similarly oriented* but if these senses be opposite they are said to be *oppositely oriented.*

If two directed lines be parallel and the senses of both are in the direction of parallelism or opposite to it, they are said to be *similarly*

* For a summary of the various cases that arise see the paper referred to in the introduction.

oriented, but if one of the senses be in the direction of parallelism and the other against it they are said to be *oppositely oriented.*

Two directed lines with a common perpendicular are called *similarly oriented* if they have the same sense relative to a point on this common perpendicular produced, while they are said to be *oppositely oriented* if their senses relative to such a point are opposite.

A directed element is said to be intimate with a basic element when the base of the former is intimate with the latter.

Two directed elements are said to be intimate when their bases are intimate.

3. DIVERGENCE.

The *divergence* between the two directed points at a distance d apart is measured by $-\cosh d$ or $+\cosh d$ according as the points are similarly or oppositely oriented.

The *divergence* between a directed point and a directed line at a distance d from it is measured by $-\sinh d$ or $+\sinh d$ according as the point and the line are similarly or oppositely oriented.

The *divergence* between two directed lines meeting at a point and making an angle δ with one another is measured by $\cos \delta$.

The *divergence* between two directed lines parallel to one another is measured by $+1$ or -1 according as they are similarly or opppositely oriented.

The *divergence* between two directed lines with a common perpendicular of length d is measured by $+\cosh d$ or $-\cosh d$ according as the lines are similarly or oppositely directed.

If we denote *divergence* by *div*, then evidently we have

$$\text{div}\ (\alpha_+, \beta_+) = \text{div}\ (\alpha_-, \beta_-) = -\text{div}\ (\alpha_+, \beta_-) = -\text{div}\ (\alpha_-, \beta_+)$$

It should be noted that *the necessary and sufficient condition that two directed elements* α_0 *and* β_0 *are intimate is* $\text{div}\ (\alpha_0 \beta_0) = 0$.

4. CO-ORDINATES OF ELEMENTS REFERRED TO A SELF-INTIMATE TRIAD.

A triad of directed elements such that each is intimate with the other two, will be called a *self-intimate triad.*

Let ξ_0 and η_0 be two directed lines intimate with one another. Let ζ_0 be a directed point intimate with both ξ_0 and η_0. Then ξ_0, η_0, ζ_0 form a self-intimate triad. If α_0 be any other directed element then

$$\text{div}\ (\alpha_0\ \xi_0),\ \text{div}\ (\alpha_0\ \eta_0),\ \text{div}\ (\alpha_0\ \zeta_0)$$

will be called the *triadic coordinates* of α_0.

5. THE IDENTICAL RELATION SATISFIED BY THE COORDINATES OF A DIRECTED ELEMENT.

Case I. Let P_0 be a directed point with coordinates x_1, y_1, z_1. Let r be the length of the radius vector drawn from ζ_0 to P_0 and θ the angle which this radius vector makes with ξ_0. Also let u and v be the lengths of the perpendiculars drawn from P_0 to ξ_0 and η_0 respectively [See fig (1)]. Then

$$x_1 = \text{div } (P_0\ \xi_0) = -\sinh u = -\sinh r \sin\theta \qquad \dots \quad (1)$$

$$y_1 = \text{div } (P_0\ \eta_0) = \sinh v = \sinh r \cos\theta \qquad \dots \quad (2)$$

$$z_1 = \text{div } (P_0\ \zeta_0) = -\cosh r \qquad \dots \quad (3)$$

Hence $$x_1^2 + y_1^2 - z_1^2 = -1.$$

Case II. Again let l_0 be a directed line with coordinates x_2, y_2, z_2. Let p be the length of the perpendicular from ζ_0 on l_0 and θ the angle this perpendicular makes with ξ_0. Let ϕ and ψ be the angles which l_0 makes with ξ_0 and η_0 respectively [See fig. (2)].

Now $$x_2 = \text{div } (l_0\ \xi_0) = \cos\phi = \sin\theta \cosh p \qquad \dots \quad (4)$$

$$y_2 = \text{div } (l_0\ \eta_0) = \cos\psi = \cos\theta \cosh p \qquad \dots \quad (5)$$

$$z_2 = \text{div } (l_0\ \zeta_0) = -\sinh p \qquad \dots \quad (6)$$

Hence $$x_2^2 + y_2^2 - z_2^2 = +1.$$

If x, y, z be the co-ordinates of a directed element

$$x^2 + y^2 - z^2 = \mp 1 \qquad \dots \quad (7)$$

the upper or the lower sign being taken according as the element is a point or a line.

6. FUNDAMENTAL THEOREM.

If x_1, y_1, z_1 be the co-ordinates of a directed element α_0 and x_2, y_2, z_2 the co-ordinates of a directed element β_0, then

$$\text{div } (\alpha_0\ \beta_0) = x_1 x_2 + y_1 y_2 - z_1 z_2 \qquad \dots \quad (8)$$

Case I. Let α_0, β_0 be similarly directed points. Let r_1, r_2 be the lengths of the radius vectors from ζ_0 to α_0 and let θ_1, θ_2 be the angles which these radius vectors makes with ξ_0. Also let d be the distance between α_0 and β_0. Then [See fig. (3)]

$$x_1=-\sinh r_1 \sin\theta_1, \qquad x_2=-\sinh r_2 \sin\theta_2 \quad \text{from} \quad (1)$$

$$y_1=\sinh r_1 \cos\theta_1, \qquad y_2=\sinh r_2 \cos\theta_2 \quad \text{from} \quad (2)$$

$$z_1=-\cosh r_1, \qquad z_2=-\cosh r_2 \quad \text{from} \quad (3)$$

Therefore
$$\begin{aligned} x_1x_2+y_1y_2-z_1z_2 &= \sinh r_1 \sinh r_2 \cos(\theta_1-\theta_2) - \cosh r_1 \cosh r_2 \\ &= -\cosh d \\ &= \operatorname{div}(\alpha_0\, \beta_0) \end{aligned}$$

The same result would be seen to hold when α_0, β_0 are oppositely directed.

Case II. Let α_0, β_0 be directed lines. Let p_1, p_2 be the lengths of the perpendiculars drawn from ζ_0 to α_0 and β_0 respectively and let θ_1, θ_2 be the angles which these perpendiculars make with ξ_0. Also let δ be the angle between α_0 and β_0, Then [See Fig. (4)]

$$x_1=-\sin\theta_1 \cosh p_1, \quad x_2=-\sin\theta_2 \cosh p_2 \quad \text{from} \quad (5)$$

$$y_1=\cos\theta_1 \cosh p_1, \quad y_2=\cos\theta_2 \cosh p_2 \quad \text{from} \quad (6)$$

$$z_1=-\sinh p_1, \qquad z_2=-\sinh p_2 \quad \text{from} \quad (7)$$

Therefore
$$\begin{aligned} x_1x_2+y_1y_2-z_1z_2 &= \cosh p_1 \cosh p_2 \cos(\theta_1-\theta_2) - \sinh p_1 \sinh p_2 \\ &= \cos\delta \end{aligned}$$

The same result would be seen to hold when α_0, β_0 are parallel or non-intersecting.

Case III. Let α_0 be a directed point and β_0 a directed line. This case can be treated on lines similar to those adopted before.

7. Equation of a basic element.

We shall show that the co-ordinates of all directed elements intimate with a given basic element (point, line or horocycle) satisfy a linear equation. This equation will be called the *triadic equation* of the given element.

Case I. Let the given basic element a be a point or a line.

Let a_0 be an orient of a. Let a, b, c be the co-ordinates of a_0. Let γ_0 be any directed element intimate with a, and let x, y, z the co-ordinates of γ_0. Hence by definition γ_0 and a_0 are intimate. Thus div $(a_0\gamma_0)$ vanishes. We then get from (8)

$$ax+by-cz=0 \qquad \dots \quad (9)$$

The linear equation (9) is satisfied by the co-ordinates of all directed elements intimate with a.

Corollary (1) : If $ax+by-cz=0$ be the equation of a, a basic point or a basic line, and p, q, r are the co-ordinates of a_0, an orient of a, then

$$\frac{p}{a}=\frac{q}{b}=\frac{r}{c} \qquad \dots \quad (10)$$

Corollary (2) : If $ax+by-cz=0$ be the equation of a point, we have

$$a^2+b^2<c^2 \qquad \dots \quad (11)$$

but if the same is the equation of a line

$$a^2+b^2>c^2 \qquad \dots \quad (12)$$

This result follows from (7) and (10).

Case II. Let the given element a be a horocycle.

Let p_1, q_1, r_1 and p_2, q_2, r_2 be the co-ordinates of two fixed similarly directed parallel lines β_0 and γ_0 intimate with the horocycle. Let x, y, z be the co-ordinates of an arbitrary directed line δ_0 intimate with a. Then δ_0 is parallel to both β_0 and γ_0, and is either similarly directed to both β_0 and γ_0 or is oppositely directed to both. In the former case div $(\delta_0\beta_0)$ = div $(\delta_0\gamma_0)$ = $+1$, while in the latter case div $(\delta_0\beta_0)$= div $(\delta_0\gamma_0)$=-1. Hence from (8)

$$p_1x+q_1y-r_1z=p_2x+q_2y-r_2z$$

$$\text{or } (p_1-p_2)x+(q_1-q_2)y-(r_1-r_2)z=0 \qquad \dots \quad (13)$$

The linear equation (13) is then satisfied by all directed elements intimate with a.

Corollary. If $ax+by-cz=0$ be the equation of a horocyclic element we have

$$a^2+b^2=c^2 \qquad \dots \quad (14)$$

For, $(p_1-p_2)^2+(q_1-q_2)^2-(r_1-r_2)^2$

$$=(p_1^2+q_1^2-r_1^2)+(p_2^2+q_2^2-r_2^2)$$
$$-2(p_1p_2+q_1q_2-r_1r_2)$$
$$=1+1-2$$
$$=0$$

8. THE CONDITION OF INTIMACY OF TWO ELEMENTS WHOSE EQUATIONS ARE GIVEN.

Theorem. If $a_1x+b_1y-c_1z=0$ *and* $a_2x+b_2y-c_2z=0$ *be the equations of two basic elements* α *and* β, *the necessary and sufficient condition that* α *and* β *are intimate is*

$$a_1a_2 + b_1b_2 - c_1c_2 = 0 \qquad \ldots \quad (15)$$

Case I. Let neither of α and β be horocyclic.

Let α_0 be an orient of α and β_0 an orient of β. Let p_1, q_1, r_1 be the co-ordinates of α_0 and p_2, q_2, r_2 the co-ordinates of β_0, then from (10)

$$\frac{a_1}{p_1} = \frac{b_1}{q_1} = \frac{c_1}{r_1} = k_1 \text{ (say)}$$

and $$\frac{a_2}{p_2} = \frac{b_2}{q_2} = \frac{c_2}{r_2} = k_2 \text{ (say)}$$

Therefore $a_1a_2+b_1b_2-c_1c_2=k_1k_2\,(p_1p_2+q_1q_2-r_1r_2)$

$$=k_1k_2 \text{ div } (\alpha_0\beta_0)$$

This shows that the necessary and sufficient condition for the intimacy of α_0 and β_0, and hence of α and β is

$$a_1a_2+b_1b_2-c_1c_2=0$$

Case II. Let α be horocyclic.

If β is intimate with α, then β must either be an equivalent horocycle in which case

$$a_1a_2+b_1b_2-c_1c_2=a_1^2+b_1^2-c_1^2=0 \text{ from (14)}$$

The result follows at once from the fact that the join is intimate with both the given elements.

Corollary (2): The necessary and sufficient condition that the elements a, β, γ *whose equations are*

$$a_1x+b_1y-c_1z=0$$

$$a_2x+b_2y-c_2z=0$$

$$a_3x+b_3y-c_3z=0$$

be co-intimate is the vanishing of the diterminant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \dots \quad (17)$$

9. THE EQUATION OF THE SYMMETRIC BETWEEN TWO DIRECTED ELEMENTS.

Let a_0 and β_0 be two directed elements, then there exists a unique basic element λ such that, all directed elements intimate with it are equidivergent with a_0 and β_0. λ is then defined to be the *symmetric* between a_0 and β_0.

Let p_1, q_1, r_1 ; p_2, q_2, r_2 be the co-ordinates of a_0 and β_0 respectively then the equation of λ, the symmetric between them, is

$$(p_1-p_2)x+(q_1-q_2)y-(r_1-r_2)z=0 \qquad \dots \quad (18)$$

For let x_1, y_1, z_1 be the co-ordinates of any directed element γ_0 intimate with λ, then

$$(p_1-p_2)x_1+(q_1-q_2)y_1-(r_1-r_2)z_1=0$$

or $\quad p_1x_1+q_1y_1-r_1z_1=p_2x_2+q_2y_2-r_2z_2$

or $\quad \text{div}\,(a_0\gamma_0)=\text{div}\,(\beta_0\gamma_0)$

To show that the symmetric is unique, we note that if there is any other element with equation

$$lx+my-nz=0 \qquad \dots \quad (ii)$$

which satisfies the definition of the symmetric, then

$$(p_1-p_2)x+(q_1-q_2)y-(r_1-r_2)z=0 \qquad \dots \quad (iii)$$

is satisfied for all values of x, y, z which satisfy (ii). Whence the equations (ii) and (iii) must be identical.

It has been shown in the paper referred to in the introduction that the symmetric between,

(i) *Two similarly directed points* P *and* Q *is the right bisector of* PQ.

(ii) *Two oppositely directed points* P *and* Q *is the mid-point of* PQ.

(iii) *A directed point* P *and a line* AB *similarly directed to it is the principal line* of* P *and* AB.

(iv) *A directed point* P *and a line* AB *oppositely directed to it is the principal point of* P† *and* AB.

(v) *The symmetric between two similarly directed parallel lines is a horocycle having both lines as axes.*

(vi) *The symmetric between two oppisitely directed parallel lines is their middle parallel.* ‡

(vii) *The directed lines* OA *and* OB *meeting at* O *is the external bisector of the angle* AOB.

($viii$) *Two similarly directed lines with a common perpendicular is the mid-point of this perpendicular.*

(ix) *Two oppositely directed lines with a common perpendicular is the line bisecting this perpendicular at right angles.*

(x) *A directed point* P *and a directed line* AB *intimate with it is a horocycle having as an axis the directed line PL, the sense of* AB *relative to L being the same as the sense of the directed point* P.

10. THE GENERALISED ANGLE-BISECTOR AND SIDE-BISECTOR THEOREM.

If a_0, β_0, γ_0 *be three directed elements and if* λ *be the symmetric between* β_0 *and* γ_0, μ *the symmetric between* γ_0 *and* a_0, ν *the symmetric between* a_0 *and* β_0 *then* λ, μ, ν *are co-intimate.*§

* The principal line of P and AB is defined as follows:—Draw PL perpendicular to AB meeting AB at L. Take P′ on PL such that P′L is complementary to PL, P and P′ lying on the same side of L. Let M be the mid-point of PP′. Then the line perpendicular to PP′ at M is defined to be the principal line of P and AB.

† The principal point of P and AB is defined as follows:—Draw PL perpendicular to AB meeting AB at L. Take L′ on PL such that PL′ is complementary to PL, L and L′ lying on the same side of P. Let S be the mid-point of LL′. Then S is defined to be the principal point of P and AB.

‡ The locus of points equidistant from two given parallel lines is a line parallel to both. This line is defined to be the middle parallel of the two given lines.

§ For a summary of cases see Art 12. *Loc. cit.*

Let a_1, b_1, c_1 ; a_2, b_2, c_2 ; a_3, b_3, c_3 ; be the co-ordinates of α_0, β_0, γ_0 respectively. Then the equations of the symmetrics λ, μ, ν are respectively

$$(a_2-a_3)x+(b_2-b_3)y-(c_2-c_3)z=0$$

$$(a_3-a_1)x+(b_3-b_1)y-(c_3-c_1)z=0$$

$$(a_1-a_2)x+(b_1-b_2)y-(c_1-c_2)z=0$$

Since the determinant

$$\begin{vmatrix} a_2-a_3 & b_2-b_3 & c_2-c_3 \\ a_3-a_1 & b_3-b_1 & c_3-c_1 \\ a_1-a_2 & b_1-b_2 & c_1-c_2 \end{vmatrix}$$

identically vanishes, the theorem is established.

11. The generalised median theorem.

If α_+, β_+, γ_+ be three directed elements, and λ, u, v be the symmetrics between β_+ and γ_-, γ_+ and α_-, α_+ and β_- respectively, then the basic elements $(\alpha\lambda)$, (βu), (γv) are co-intimate.

Let the co-ordinates of α_+, β_+, γ_+ be respectively a_1, b_1, c_1 ; a_2, b_2, c_2 ; a_3, b_3, c_3. Then the equation of λ is

$$(a_2+a_3)x+(b_2+b_3)y-(c_2+c_3)z=0$$

and the equation of α is

$$a_1x+b_1y-c_1z=0$$

Hence the equation of $(\alpha\lambda)$ the join of α and λ is

$$(A_2-A_3)x+(B_2-B_3)y-(C_2-C_3)z=0$$

where A_1, B_1 etc. are the minors of the corresponding small letters in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Similarly the equations of $(\beta\mu)$ and $(\gamma\nu)$ are

$$(A_3-A_1)x+(B_3-B_1)y+(C_3-C_1)z=0$$

$$(A_1-A_2)x+(B_1-B_2)y-(C_1-C_2)z=0$$

Since the deteminant

$$\begin{vmatrix} A_2-A_3 & B_2-B_3 & C_2-C_3 \\ A_3-A_1 & B_3-B_1 & C_3-C_1 \\ A_1-A_2 & B_1-B_2 & C_1-C_2 \end{vmatrix}$$

vanishes identically the theorem is established.

12. The generalised perpendicular theorem.

If α, β, γ be three basic elements, then the three elements $\{(\beta\gamma)\alpha\}$, $\{(\gamma\alpha)\beta\}$, $\{(\alpha\beta)\gamma\}$ are co-intimate.

Let

$$a_1x+b_1y-c_1z=0$$

$$a_2x+b_2y-c_2z=0$$

$$a_3x+b_3y-c_3z=0$$

be the equations of α, β, γ.

Then the equation $(\beta\gamma)$ the join of β and γ is

$$(b_2c_3-b_3c_2)x+(c_2a_3-c_3a_2)y-(a_2b_3-a_3b_2)z=0$$

or $$A_1x+B_1y-C_1z=0$$

where A_1, B_1 etc are as before.

The equation of $\{(\beta\gamma)\alpha\}$, the join of $(\beta\gamma)$ and α is then

$$(b_1C_1-c_1B_1)x+(c_1A_1-a_1C_1)y-(a_1B_1-b_1A_1)z=0$$

and similar equations may be obtained for $\{(\gamma\alpha)\beta\}$ and $\{(\alpha\beta)\gamma\}$.

Now consider the determinant

$$\begin{vmatrix} b_1C_1-c_1B_1 & c_1A_1-a_1C_1 & a_1B_1-b_1A_1 \\ b_1C_2-c_2B_2 & c_2A_2-a_2C_2 & a_2B_2-b_2A_2 \\ b_3C_3-c_3B_3 & c_3A_3-a_3C_3 & a_3B_3-b_3A_3 \end{vmatrix}$$

The sum of the constituents in the first column is

$$(b_1C_1+b_2C_2+b_3C_3)-(c_1B_1+c_2B_2+c_3B_3)$$

which is zero. Similarly the sum of the elements in every column is zero. Hence the determinant identically vanishes and this establishes our theorem.

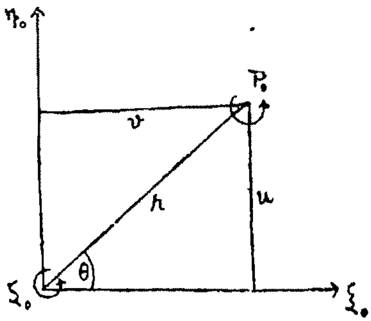

Fig (1)

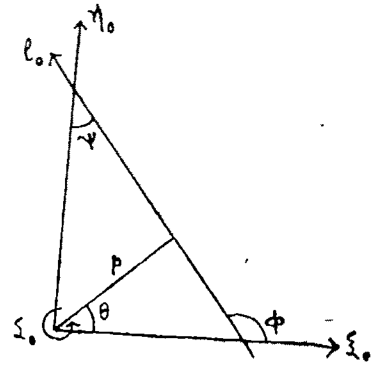

Fig (2)

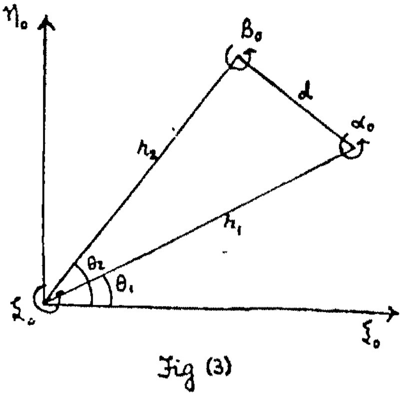

Fig (3)

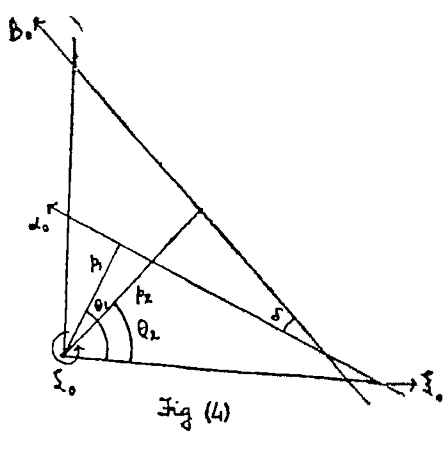

Fig (4)

On certain integral equations of the second kind and the construction of Mathieu Functions

BY

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In a paper published in the International Congress of Mathematics,* Prof. Whittaker was the first to obtain certain homogeneous integral equation connected with Mathieu Functions and had shown how they could be utilised for the construction of these functions. In a paper published in the *Journal of Science*, Vol. III, Calcutta University, I obtained a general form of the homogeneous integral equation of Whittaker's type connected with Mathieu Functions. It is the object of this paper to obtain integral equations of the second kind connected with these functions and those of the second kind and to show how they can be used to construct them.

§ 1.

Let it be required to obtain the general solutions of differential equation of the type

$$\frac{d^2y}{dz^2}+Qy=Ry, \qquad \ldots \quad (1)$$

where Q, R are functions of z.

Supose $y_1(z)$ and $y_2(z)$ are any two solutions of

$$\frac{d^2y}{dz^2}+Qy=0, \qquad \ldots \quad (2)$$

Let us now assume that

$$y(z)=u_1(z)y_1(z)+u_2(z)y_2(z), \qquad \ldots \quad (3)$$

Fifth International Congress of Mathematics, Vol. II, 1912.

is the general solution of (1), where u_1 and u_2 are two arbitrary functions of z, whose forms are to be obtained subject to the conditions

$$y_1\frac{du_1}{dz}+y_2\frac{du_2}{dz}=0. \qquad \dots \quad (4)$$

Now substituting $y(z)$ of (3) in (1) and simplifying by the help of the above relation, we obtain

$$\frac{du_1}{dz}\cdot\frac{dy_1}{dz}+\frac{du_2}{dz}\cdot\frac{dy_2}{dz}=\mathrm{R}(u_1y_1+u_2y_2) \qquad \dots \quad (5)$$

But from (4) we have

$$-\frac{\frac{du_1}{dz}}{y_2}=\frac{\frac{du_2}{dz}}{y_1}=\mathrm{Z}, \text{ (say)}$$

then $$\frac{du_1}{dz}=-\mathrm{Z}y_2, \quad \frac{du_2}{dz}=\mathrm{Z}y_1$$

and the relation (5) becomes

$$\mathrm{Z}\left(y_1\frac{dy_2}{dz}-y_2\frac{dy_1}{dz}\right)=\mathrm{R}(u_1y_1+u_2y_2), \qquad \dots \quad (6)$$

Now, y_1 and y_2 being solutions of (2), we have

$$\begin{vmatrix} y_1'' & y_1 \\ y_2'' & y_2 \end{vmatrix}=0$$

which on integration reduces to

$$y_1\frac{dy_2}{dz}-y_2\frac{dy_1}{dz}=\mathrm{C}, \qquad \dots \quad (7)$$

where C is an arbitrary constant.

Hence we have

$$\mathrm{CZ}=\mathrm{R}(u_1y_1+u_2y_2).$$

Therefore the solution of the equation (1) will be given by

$$y(z)=u_1y_1+u_2y_2,$$

when we take

$$Z=\frac{R}{C}y(z).$$

With this value of Z, we obtain the values of u_1 and u_2 as

$$(a)\quad \frac{du_1}{dz}=-\frac{R}{C}y(z).y_2,\qquad \text{or}\qquad u_1=A-\frac{1}{C}\int_a^z R(t)y(t)y_2(t)dt$$

$$(b)\quad \frac{du_2}{dz}=\frac{R}{C}y(z).y_1,\qquad \text{or}\qquad u_2=B+\frac{1}{C}\int_a^z R(t)y(t)y_1(t)dt$$

where A, B, a are arbitrary constants.

Hence the solution (3) reduces to an integral equation of the second kind as

$$y(z)=(Ay_1+By_2)+\frac{1}{C}\int_a^z R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z) \end{vmatrix} y(t)dt \quad \dots \quad (8)$$

§ 2.

On examining the relation (7) we find that the equation (2) cannot have two independent solutions which are both odd or both even. Hence of the two solutions of (2), one is odd and the other even. Let $y_1(z)$ be the odd and $y_2(z)$ the even solution. Then if R be an even function, the integral equation

$$y(z)=By_2(z)+\frac{1}{C}\int_0^z R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z) \end{vmatrix} y(t)dt, \quad \dots \quad (9)$$

will give *even* solutions of (1). This can be seen as follows:—

Writing $-z$ for z in (9), we get

$$y(-z)=By_2(-z)+\frac{1}{C}\int_0^{-z} R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(-z) & y_2(-z) \end{vmatrix} y(t)dt$$

$$=By_2(z)-\frac{1}{C}\int_0^z R(-t)\begin{vmatrix} y_1(-t) & y_2(-t) \\ y_1(-z) & y_2(-z) \end{vmatrix} y(-t)dt$$

$$=By_2(z)+\frac{1}{C}\int_0^z R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z) \end{vmatrix} y(-t)dt$$

which is of the same form as relation (9). Therefore $y(z)$ and $y(-z)$ satisfy the same integral equation. Now as the solution of an integral equation of the second kind is unique, we must have

$$y(-z)=y(z)$$

or in other words, the relation (9) will give even solutions of (1).

Similarly the odd solutions will be given by the integral equation,

$$y(z)=Ay_1(z)+\frac{1}{C}\int_0^z R(t)\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z) \end{vmatrix} y(t)dt \qquad \ldots \quad (10)$$

§ 3.

Let us now apply the above method to find the solutions of Mathieu's equation

$$\frac{d^2y}{dz^2}+(A+16q\cos 2z)y=0,$$

which can, for the present purpose, be written in the form

$$\frac{d^2y}{dz^2}+m^2y=-(a'+16q\cos 2z)y, \qquad \ldots \quad (11)$$

where $A=m^2+a'$, a' being an arbitrary constant.

The even solutions of the above equations will be given, for different values of m and a', by the integral equation

$$y(z)=B\cos mz+\frac{1}{m}\int_0^z (a'+16q\cos 2t)\sin m(t-z)y(t)dt \ldots \quad (12)$$

Suppose

$$y(z)=f_0(z)+qf_1(z)+q^2f_2(z)+\dots$$

where f_0, f_1 ... etc. are periodic functions of z only and do not contain q, and suppose also

$$a'=a_1q+a_2q^2+\dots\text{etc.},$$

where a_1, a_2,...are numerical constants; and

$$B=1+b_1q+b_2q^2+\dots\text{etc.},$$

where b_1, b_2,... are numerical constants.

On substituting in the integral equation, we find

$$f_0(z)+qf_1(z)+q^2f_2(z)+\dots=(1+b_1q+b_2q^2+\dots)\cos mz$$

$$+\frac{1}{m}\int_0^z\{(a_1q+a_2q^2+\dots)+16q\cos 2t\}$$

$$\times\sin m(t-z)\{f_0+qf_1(t)+\dots\}dt.$$

Hence equating the co-efficients of successive powers of q, we get

(i) $f_0(z)=\cos mz$

(ii) $f_1(z)=b_1\cos mz+\dfrac{1}{m}\displaystyle\int_0^z(a_1+16\cos 2t)\sin m(t-z)f_0(t)dt,$

(iii) $f_2(z)=b_2\cos mz+\dfrac{a_2}{m}\displaystyle\int_0^z\sin m(t-z)f_0(t)dt$

$$+\frac{1}{m}\int_0^z(a_1+16\cos 2t)\sin m(t-z)f_1(t)dt,$$

... ...

The expressions $f_1(z), f_2(z)$,...etc., will have to be obtained in succession from the above relation, subject to the condition that a_1, a_2, ...must be such as will give f_1, f_2,......, etc., periodic forms. The values of b_1, b_2,...will be obtained from the condition that the expressions for f_1, f_2,... will not contain $\cos mz$ as a term in them.

Thus on integrating, we obtain :—

$$(a)\quad f_1(z)=\left(b_1+\frac{4}{m^2-1}\right)\cos mz-\frac{a_1}{2m}z\sin mz$$

$$+\left\{\frac{2\cos(m+2)z}{m+1}-\frac{2\cos(m-2)z}{m-1}\right\}.$$

$\therefore$ if $a_1=0$, and $b_1=-\frac{4}{m^2-1}$, then

$$f_1(z)=\frac{2\cos(m+2)z}{m+1}-\frac{2\cos(m-2)z}{m-1}$$

$$(b)\quad f_2(z)=\left\{b_2-\frac{4(m^2+2)}{(m^2-1^2)(m^2-2^2)}\right\}\cos mz$$

$$+z\sin mz\left\{\frac{-a_2}{2m}+\frac{16}{m(m^2-1)}\right\}$$

$$+\left\{\frac{2\cos(m-4)z}{(m-1)(m-2)}+\frac{2\cos(m+4)z}{(m+1)(m+2)}\right\}.$$

$\therefore$ if $a_2=\frac{32}{m^2-1}$, and $b=\frac{4(m^2+2)}{(m^2-1^2)(m^2-2^2)}$, then

$$f_2(z)=\frac{2\cos(m-4)z}{(m-1)(m-4)}+\frac{2\cos(m+4)z}{(m+1)(m+2)}$$

and so on.

Thus we see that when

$$A=m^2+\frac{32q^2}{m^2-1}+\ldots\text{etc.}$$

the solution of Mathieu's equation as obtained by the integral equations (11) is

$$\cos mz+q\left\{-\frac{2\cos(m-2)z}{m-1}+\frac{2\cos(m+2)z}{m+1}\right\}$$

$$+q^2\left\{\frac{2\cos(m-4)z}{(m-1)(m-2)}+\frac{2\cos(m+4)z}{(m+1)(m+2)}\right\}+\ldots\text{etc.}$$

which is denoted by $ce_m(z, q)$.

§ 4.

We will obtain by the above method the solution $ce_1(z, q)$, a particular case of the above. We obtain the expressions of $f_0, f_1, f_2, \cdot$ etc., as given below :—

(*i*) $f_0(z)=\cos z$

(*ii*) $f_1(z)=b_1 \cos z+\int_0^z (a_1+16\cos 2t)\sin(t-z)\cos t\,dt$

$$=(b_1-1)\cos z-(4+\tfrac{1}{2}a_1)z\sin z+\cos 3z$$

$\therefore$ if $a_1=-8$, $b_1=1$ and $f_1(z)=\cos 3z$

(*iii*) $f_2(z)=b_2\cos z+a_2\int_0^z \sin(t-z)\cos t\,dt$

$$+\int_0^z (a_1+16\cos 2t)\sin(t-z)\cos 3t\,dt$$

$$=(b_2+\tfrac{2}{3})\cos z-(\tfrac{1}{2}a_2+4)z\sin z+\tfrac{1}{3}\cos 5z-\cos 3z$$

$\therefore$ if $a_2=-8$, $b_2=-\frac{2}{3}$, $f_2(z)=\frac{1}{3}\cos 5z-\cos 3z$

$$(iv)\quad f_3(z) = b_3 \cos z + a_3 \int_0^z \sin(t-z)\cos t\,dt$$

$$+ a_2 \int_0^z \sin(t-z)\cos 3t\,dt$$

$$+ \int_0^z (a_1 + 16\cos 2t)\sin(t-z) f_2(t)\,dt$$

$$= (b_3 + \tfrac{1}{18})\cos z + \left(4 - \frac{a_3}{2}\right) z \sin z + \tfrac{1}{8}\cos 3z$$

$$- \tfrac{4}{9}\cos 5z + \tfrac{1}{18}\cos 7z$$

$\therefore$ if $a_3 = 8$, and $b_3 = -\frac{1}{18}$, then

$$f_3(z) = \tfrac{1}{8}\cos 3z - \tfrac{4}{9}\cos 5z + \tfrac{1}{18}\cos 7z$$

and so on.

Hence when

$$A = 1 - 8q - 8q^2 + 8q^3 + \ldots \text{etc.},$$

the solution of the integral equation is

$$\cos z + q\cos 3z + q^2(\tfrac{1}{3}\cos 5z - \cos 3z)$$

$$+ q^3(\tfrac{1}{8}\cos 3z - \tfrac{4}{9}\cos 5z + \tfrac{1}{18}\cos 7z) + \ldots \text{etc.},$$

which is nothing but the solution $ce_1(z, q)$.

N.B.—The solution $ce_1(z, q)$ can also be obtained as the solution of the integral equation of indirect integral form as

$$y(z) = \cos z + \int^z (a' + 16q\cos 2t)\sin(t-z)\,y(t)\,dt,$$

where $\qquad a' = -8q - 8q^2 + 8q^3 - \ldots \text{etc.}$

§ 5.

The construction of the solutions of the second kind.

The integral equations which we have obtained above, can also be used to construct the solutions of the second kind of Mathieu's equation (11). This method, though sometimes laborious, is yet useful. We will first proceed to obtain the solution $i\eta_m(z, q)$ by means of the integral equations

$$y(z)=\beta \sin mz+\frac{1}{m}\int_0^z (a'+16q \cos 2t) \sin m(t-z)y(t)dt \quad \ldots \quad (13)$$

where

$$\beta=1+b_1q+b_2q^2+\ldots$$

and

$$a'=\frac{32q^2}{m^2-1}-\frac{128(5m^2+7)q^4}{(m^2-1)^3(m^2-4)} - \text{etc.}$$

$$=a_2q^2-a_4q^4\ldots\text{etc.},$$

this expression for a' being the same as was obtained in § 3 for obtaining the solution $ce_m(z, q)$ from the integral equation (12).

Now suppose that

$$y(z)=f_0(z)+qf_1(z)+q^2f_2(z)+\ldots\ldots\ldots\ldots\text{etc.},$$

where $f_0, f_1\ldots$ are functions of z only and do not contain q.

Substituting in the integral equation (13) and equating like powers of q, we get

(*i*) $f_0(z)=\sin mz$

(*ii*) $f_1(z)=b_1 \sin mz + \dfrac{1}{m}\displaystyle\int_0^z 16 \cos 2t \sin m(t-z)f_0(t)\, dt$

(*iii*) $f_2(z)=b_2 \sin mz +\dfrac{a_2}{m}\displaystyle\int_0^z \sin m(t-z)f_0(t)dt$

$$+\frac{1}{m}\int_0^z 16 \cos 2t \sin m(t-z)f_1(t)dt,$$

... ...

On solving the above equations in succession, we obtain $f_1, f_2, \ldots$ etc. The numerical constants $b_1, b_2, \ldots$ are obtained from the condition that the forms of $f_1, f_2, \ldots$ must not contain $\sin mz$ as a term in them. Thus we obtain

$$(a) \qquad f_1(z) = \left(b_1 - \frac{4}{m^2-1}\right)\sin mz + \frac{2\sin(m+2)z}{m+1} - \frac{2\sin(m-2)z}{m-1}$$

$$\therefore \text{ if } b_1 = \frac{4}{m^2-1}, \text{ then}$$

$$f_1(z) = \frac{2\sin(m+2)z}{m+1} - \frac{2\sin(m-2)z}{m-1}$$

$$(b) \qquad f_2(z) = \frac{2}{(m+1)(m+2)}\sin(m+4)z + \frac{2}{(m-1)(m-2)}\sin(m-4)z,$$

where
$$b_2 = \frac{4(m^4+4m^3-14m^2-16m+16)}{m^2(m^2-1^2)(m^2-2^2)}$$

and so on.

Proceeding as above we will obtain the solution $i\eta_m(z, q)$. We proceed to show the method by working out a few particular cases.

(*a*) Let us construct the integral $i\eta_1(z, q)$ by means of the integral equations (13). To obtain the forms of $f_0, f_1, \ldots$ we shall have to solve the following set of equations :—

(*i*) $f_0(z) = \sin z$

$$(ii) \quad f_1(z) = b_1 \sin z + \int_0^z (-8+16\cos 2t)\sin(t-z) f_0(t)\,dt$$

$$(iii) \quad f_2(z) = b_2 \sin z - 8\int_0^z \sin(t-z) f_0(t)\,dt$$

$$+ \int_0^z (-8+16\cos 2t)\sin(t-z) f_1(t)\,dt$$

... ...

Solving the above relations in succession, we get

(a) $f_0(z)=\sin z$

(b) $f_1(z)=b_1 \sin z-8z \cos z+\sin 3z+5 \sin z,$

Now, choose $b_1=-5$, so that $f_1(z)$ does not contain $\sin z$. then

$$f_1(z)=\sin 3z-8z \cos z.$$

(c) $f_2(z)=(b_2-\tfrac{25}{3})\sin z+5 \sin 3z+\tfrac{1}{3} \sin 5z-8z \cos 3z$

$\therefore$ if $b_2=\tfrac{25}{3}$, then $f_2(z)=(\tfrac{1}{3} \sin 5z+5 \sin 3z)-8z \cos 3z$

(d) $f_3(z)=(b_3-8\tfrac{1}{18})\sin z+24z \cos z+8z \cos 3z-\tfrac{8}{3}z \cos 5z$

$$+\tfrac{1}{18} \sin 7z+\tfrac{8}{3} \sin 5z-\tfrac{25}{3} \sin 3z.$$

$\therefore$ if $b_3=8\tfrac{1}{18}$, then

$f_3(z)=z(24 \cos z+8 \cos 3z-\tfrac{8}{3} \cos 5z)+(\tfrac{1}{18} \cos 7z+\tfrac{8}{3} \sin 5z-\tfrac{25}{3} \sin 3z)$

Hence is $i\eta_1(z, q)$ is given by the integral equation

$$y(z)+(1-5q+\tfrac{25}{3}q^2+8\tfrac{1}{18}q^3+\dots) \sin z$$

$$+\int_0^z (a'+16q \cos 2t)\sin(t-z)y(t)\, dt,$$

where

$$a'=-8q-8q^2+8q^3-\dots$$

N.B.—The same integral is $i\eta_1(z, q)$ can also be obtained by solving the integral equation of indefinite integral form as

$$y(z)=(1-4q+12q^3+\dots)\sin z+\int^z (a'+16q \cos 2t)\sin(t-z)y(t)\, dt,$$

a' having the same value as above.

(b) If we proceed to construct the integral $i\eta_2(z, q)$ from the integral equation

$$y(z)=\beta \sin 2z+\tfrac{1}{2}\int_0^z (a'+16q \cos 2t)\sin 2(t-z)y(t)dt,$$

where

$$a'=\tfrac{80}{3}q^2-\tfrac{6104}{27}q^4+\ldots,$$

we obtain according to the above procedure,

(i) $f_0(z)=\sin 2z$

(ii) $f_1(z)=(b_1-\frac{4}{3})\sin 2z+\frac{2}{3}\sin 4z$

$\therefore$ if $b_1=\frac{4}{3}$, $f_1(z)=\frac{2}{3}\sin 4z$.

(iii) $f_2(z)=(b_2-\frac{8}{3})\sin 2z+8z\cos 2z+\frac{1}{6}\sin 6z$

$\therefore$ if $b_2=\frac{8}{3}$, $f_2(z)=8z\cos 2z+\frac{1}{6}\sin 6z$

(iv) $f_3(z)=(b_3+9\frac{88}{135})\sin 2z-16z+\frac{16}{3}z\cos 4z+\frac{1}{45}\sin 8z-\frac{88}{27}\sin 4z$

$\therefore$ if $b_3=-9\frac{88}{135}$,

$f_3(z)=-16z+\frac{16}{3}z\cos 4z+\frac{1}{45}\sin 8z-\frac{88}{27}\sin 4z$.

Hence

$$y(z)=\sin 2z+\tfrac{2}{3}q\sin 4z+q^2(8z\cos 2z+\tfrac{1}{6}\sin 6z)$$
$$+q^3(-16z+\tfrac{16}{3}z\cos 4z+\tfrac{1}{45}\sin 8z-\tfrac{88}{27}\sin 4z)+\ldots\ldots$$

which is nothing but the integral $i\eta_2\ (z, q)$.

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II

On the Indian Method of Root Extraction

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Introduction

In his *History of Mathematics* which has recently come out Professor D. E. Smith has made the statement that Bhāskarāchārya's method of extracting the square root of a number is similar to that given by Theon of Alexandria, and that the Hindus obtained this method from the Greeks through the Arabs.* Further on in the same chapter is given an illustration of the method of extracting square roots given by Cataneo (1546) with the remark that "among the early writers to take an important step towards our present method was Cataneo." In the course of the present paper it will be shown that Bhāskarāchārya's method of extracting square roots is the same as Cataneo's, and that it was given by the Indian mathematician Āryabhaṭa as early as 499 A.D. On the other hand, the method given by Theon of Alexandria does not in the least resemble Cataneo's method. Thus the first statement of Professor Smith is contradicted by the second. It seems that he has fallen into this error by following Mr. G. R. Kaye too closely. The blunders of Kaye have also been repeated by Professor F. Cajori.†

Kaye's mistake

Kaye has published two papers, "Notes on Indian Mathematics I—Numeral Notation", and "Notes on Indian Mathematics II—Āryabhaṭa"

* D. E. Smith, *History of Mathematics*, Vol. II (1925), pp. 144-48.

† F. Cajori, "The controversy on the origion of our numerals," *The Scientific Monthly*, Vol IX (1917), p. 450.

in the *Journal of the Asiatic Society of Bengal*, Vol. III (1907) and Vol. IV (1908) respectively. These papers besides many other errors and misrepresentations contain the mistaken assertions that *the rules for the extraction of square and cube roots given by Āryabhaṭa are perfectly general (i.e., algebraical in character) and apply to all arithmetical notations*, and that *the method was known to the Greeks and admirably expressed by Theon of Alexandria.*

The object of the present paper is to clear the mis-understanding created by the writings of these authors, and to examine closely how far can India lay claim to the invention of the present methods of extracting square and cube roots.

I

Āryabhaṭa's rule for the extraction of square roots

1. The first Indian writer to give a method for the extraction of the square root of a number was Āryabhaṭa (born 476 A.D.). In his *Āryabhaṭīya* which was composed in 499 A.D. at Kusumapura (the modern Patna), there occurs the following rule:

भागं हरेद्वर्गान्नित्यं द्विगुणेन वर्गमूलेन ।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम् ॥

(*Gaṇita-pāda, 4*)

Kaye's wrong translations

Kaye has given two translations* of this couplet. Thy are:

(1) "Always divide *the part that is not square* by twice the root of the *square*, after having subtracted from the *squared part* the *square of the root*: the quotient is the *root* to the next term."

(2) "Square having been subtracted from *square*, always the *non-square* must be divided by double the square root. The quotient placed at the next place; this gives the square root."

Taking the first of these translations, we see that the first operation to be performed is '*to subtract from the squared part the square of the root.*' Now supposing we are to find the square root of, say, 625, how are we

* Kaye, 'Notes on Indian Mathematics I & II" *Journ, Asiatic Soc. of Bengal*, Vols. III & IV respectively.

going to apply the rule? Where is *the squared part* and where is the *root* of which the square is to be subtracted? How are we to know what part of 625 is the '*squared part*' and what part of it is the '*part that is not square*'? Taking the second translation we are confronted with a similar difficulty. How are we going to subtract '*square*' from '*square*'? It is difficult under these circumstances to guess what Kaye really means by these translations.

Thus according to Kaye's translations, Āryabhaṭa's rule for the extraction of the square root of a number appears to be an array of words without any sense. But in fact, it is most certainly not so. The commentators of Āryabhaṭa, such as Sūryadeva and Parameśvar, have explained the rule so very lucidly that there can be no doubt about its true meaning. Only the incorrect translations of Kaye have made it appear simply miserable. It seems that either the learned author of these translations is profoundly ignorant of Sanskrit or the translations have been deliberately made meaningless in order to deceive unwary readers.

The correct literal rendering of the couplet ought to run as follows:

Always divide the even* place by twice the square root, after having subtracted from the odd† place the square (of the quotient); the quotient placed at the next place (gives) the root.

Illustrations‡

The digits of the number whose square root is to be found are marked off as standing in odd or even places by putting over them vertical and horizontal lines thus:

$$\overset{|}{5}\;\bar{4}\;\overset{|}{7}\;\bar{5}\;\overset{|}{6}$$

* युग्मस्थानान्यवर्गसंज्ञितानि ।

i.e. 'Avarga' is the name of the even places (*Parm*).

† ओजस्थानानि वर्गसंज्ञितानि ।

i.e. 'Varga' is the name of odd places. (*Parm.*)

‡ For an illustration of the method carried out on a '*pati*' see page 129, footnote.

and the method proceeds as below: *

		| – | – | 5 4 7 5 6 (	root=2
Subtract square†	...	4	
Twice root, *i.e.*, 2 × 2 is	4)	1 4 (3	quotient Placing quotient at the next place gives the root 23
		1 2	
		2 7	
Square of quotient	...	9	
Twice root is ... 2 23 =	46)	1 8 5 (4	quotient Placing quotient at the next place gives the root 234
		1 8 4	
		1 6	
Square of quotient	.	1 6	

Thus the required square root is 234.

Āryabhaṭa's method the same as Cataneo's

Now the method of Cataneo, which according to Smith is the forerunner of the present method, has been illustrated by him as follows: ‡

5 4 7 5 6	(234
4	Primo 4
1 4	
	Secondo 46
1 2	
2 7	
9	
1 8 5	
1 8 4	
1 6	
1 6	

* A similar division of the notational places into 'odd' and 'even' has been conceived by Āryabhaṭa in expounding his system of alphabetic numeral notation.

† 'अन्त्याद्वर्गस्थानाद्वाथा त्यक्त्वा वर्गं विशोधयेत् ।'

i.e., subtract the greatest square number from the last odd place (*Parm.*).

‡ Smith, *History of Mathematics*, Vol. II, p. 148.

A comparison of the above with the illustration of Āryabhaṭa's method given before shows that both the methods are identical, all the steps of the process being the same in the two cases.

Theon's Method

Not content with giving wrong and meaningless translations of Āryabhaṭa's rule, Kaye has made the mistaken assertain that *the method was known to the Greeks and admirably expressed by Theon of Alexandria.* Smith and Cajori have followed suit in making similar statements as already pointed out in the introduction.

Now the method given by Theon of Alexandria is as follows:

"Find the root of the nearest integer to the whole number, subtract the square, convert the remainder into primes and divide it by the double of the first root, and thus determine the next term in the root, square the sum of the terms found, subtract this square, convert the remainder into seconds and divide it by the double of the root already found, and *you will have the square root very nearly.*"

The above rule has been illustrated by finding the square root of 4500 to be 67 4′ 55″ approximately. Theon's rule differs from Catanso's or Āryabhaṭa's method in the following important particulars:

(1) It is a method of finding the *approximate** square root of a number in sexagesimal fractions, if it is not a perfect square.

(2) The greatest number whose square equals the nearest integer is to be found out by some method which has not been mentioned. Theon gives no rule for finding out the whole number of the root, 67 in this case.

(3) No mention is made of division into periods (odd and even places). The method, unlike that of Āryabhaṭa, is algebraical in character and applies to all possible notations.

Another example of finding the square root of a number is found in Theon's commentary on the *Syntaxis.* Here the square root of 144 is found out, the work being illustrated by a diagram. Theon takes an

* The detailed workings of the process have been given by Heath (*History*, Vol. I, pp. 61-63) where it has also been pointed out that "Theon's plan does not work conveniently so far as the determination of the first fractional term (the first-sixtieths) is concerned, unless the integral term in the square root is large relatively to it."

arbitrary * number 10 whose square is less than 144, and shows that after subtracting $10^2+2.10.2+2^2$ there is no remainder, and thus concludes that the square root of 144 is 12, It must be, however, noted that we may have taken any other arbitrary number, say 9 or 11, and by proceeding in the same manner got the required result. Gow in his *History of Greek Mathematics* (p. 55), after giving the above example has explicitly stated that the Greeks did not possess a definite method for the extraction of square roots. Further it may be noted that in no work of the Greeks is a method for finding out the exact square root of a large square number given. The above trivial example given by Theon can, therefore, be considered only as an illustration of the Euclidean formula $a^2+2ab+b^2=(a+b)^2$, rather than as an illustration of any definite method that might have been possessed by the Greeks.

All this makes it abundantly clear that Theon's method does not resemble the method given by Āryabhaṭa, and that the ancient Greeks were never in possession of any definite method for the extraction of square roots,† for had it been so, the method would have appeared in the writings of early Greek or European writers long before Cataneo (1546).

Rules for the extraction of square root given by other Indian Mathematicians

That a novel or forced interpretation has not been put on Āryabhaṭa's rule is evident from the writings of later Indian mathematicians, who have all given the same rule. The different styles of expressing the same thing by different writers is interesting.

Śrīdhara ‡ (*c.* 750) gives the rule as follows:

विषमात् पदतस्त्यक्त्वा वर्गं स्थानच्युतेन मूलेन ।
द्विगुणेन भजेच्छेषं लब्धं विनिवेशयेत् पंक्त्याम् ॥
तद्वर्गं संशोध्य द्विगुणी कुर्वीत पूर्वं लब्धं यत् ।
उत्सार्य ततो विभजेच्छेषं द्विगुणी कृतं दलयेत् ॥

i.e., Having subtracted the square from the odd place, divide the next (even) place by twice the root which has been separately placed

* Heath in his *History of Greek Mathematics* overlooks the word 'arbitrary' and tries to conclude that the Greeks might have been in possession of a method of extracting square roots resembling our present method, but his reasoning is unconvincing.

† This conclusion is further strengthened by the fact that the Greeks had no method for finding out cube roots, which is admitted by all historians.

‡ *Triśatikā*, ed. by S. Dvivedi, p. 5.

(in a line), and after having subtracted the square of the quotient write it down in the line; double what has been obtained above (by placing the quotient in the line) and taking this divide with it the next (even) place. Halve the doubled quantity (to get the root).*

* The orthodox Hindu method of carrying out the operation on a *pāṭi*—an wooden board on which sand is spread and figures are written in the sand with the fingers—is given below:

The given number is written down on the *pāṭi* and the odd and even places are marked by vertical and horizontal lines thus

| – | – |
5 4 7 5 6

Beginning with the last odd place, the square number is subtracted. 4 subtracted from 5 gives 1. Then the number 5 is rubbed out and the remainder 1 written in its place. Thus after the first operation is performed, what stands on the *pāṭi* is

– | – |
1 4 7 5 6

The root 2 is permanently placed at a separate portion of the *pāṭi* which has been termed *pankti* (*i.e.*, line). Dividing the number up to the next even mark by twice the root, (*i.e.*) dividing 14 by 4 we obtain the quotient 3. The number 14 is now rubbed out and the remainder 2 written in its place; thus on the *pāṭi* we have now

| – |
2 7 5 6

The square of the quotient, *i.e.*, 9 is now substracted from the figures upto the next odd mark, *i.e.*, 27. This gives 18; 27 is rubbed out and 18 written in its place. The number on the *pāṭi* is now

– |
1 8 5 6

Placing the quotient in the line gives 23. Doubling the number in the line, *i.e.*, doubling 23 we get 46, and dividing by it the number upto the next even mark (*i.e.*, 185) we get the quotient 4. The remainder is 1. 185 is rubbed out and the remainder 1 written in its place. The number on the *pāṭi* now becomes

|
1 6

Substracting the square of the quotient, *i.e.*, 4^2 from the number upto the next odd mark we get no remainder. Putting the quotient in the line we get 134, and doubling it we obtain 468. But the given number on the *pāṭi* has been totally rubbed out and there remains nothing on which to continue the process. The doubled quantity is 468. 468 being halved gives 134 which is the required root.

A similar method of extracting square root has been given by the Arab Mathematician Ali Ibn Ahmed Al-Nasawi (1030 A.D.) who wrote a book on the Indian Calculation in Persian and subsequently in Arabic at the request of Charaf Almoulouq (see H. Suter, "Uber das Rechenbuch des Ali bin Ahmed el-Nasawi," *Bibliotheca Mathematica*, VII (3), p. 114, and F Woepcke, "Memoire sur la propagation des chiffres indiens," *Journ. Asiatique*, Tome I (6), 1863.

Mahāvirāchārya (c. 850) expresses the rule thus : *

अन्त्यौजादपहृत कृति मूलेन द्विगुणितेन युग्म हृतौ ।
लब्धं कृतिस्त्याज्यौजे द्विगुण दलं वर्गमूलफलम् ॥

i.e., From the last odd place subtract the square number ; then multiply the root by two and divide with this the even place : and then the square of the quotient is to be subtracted from the odd place. The half of the doubled quantity the resulting square root.

Āryabhaṭa II (c. 950) gives the rule thus : †

विषम समे स्थाने स्तो विषमादुपरित्यजेद्वर्गम् ।
उत्सारित मूलेन द्विगुणेन भजेत फलं न्यसेत पंक्त्याम् ॥
लब्ध कृतिं त्यक्त्वोपरि जह्याद्द्विगुणं दलीकृतं मूलम् ।

i.e, *Viṣama* (odd) and *sama* (even) are the (names of the) places. From the last odd place subtract the square number, then divide the next place by twice the root (of the above square) which has been placed in a separate line, and place the quotient in the (same) line ; then subtract the square of the quotient from the place next to the one from which the quotient has been obtained. Half of the doubled quantity is the root.

Bhāskarāchārya (b. 1114) gives the rule thus : ‡

त्यक्त्वान्त्याद्विषमात्कृतिं द्विगुणयेन्मूलं समेतद्धृते ।
त्यक्त्वा लब्ध कृतिं तदाद्य विषमाल्लब्धं द्विनिघ्नं न्यसेत् ॥
पंक्त्यां पंक्तिहृते समेन्य विषमात् त्यक्त्वाप्तवर्गं फलम् ।
पंक्त्यां तद्द्विगुणं न्यसेदिति मुहुः पंक्तेर्दलं स्यात्पदम् ॥

i.e., Subtract from the last odd place the greatest square number. Set down double the root in a line, and after dividing by it the next even place subtract the square of the quotient from the next odd place and set down double the quotent in the line. Thus repeat the operation

* *Gaṇitasāra Saṃgrah*, ed. by M Rangāchārya, p. 13.

The translation given below is Rangāchārya's. An illustration has also been given by him.

† *Mahāsiddhānta*, ed. by S. Dvivedī, p. 145.

‡ *Cf.* Taylor's *Lilawati*, p. 1. The method has also been illustrated by working out an example.

throughout all the figures. The half of the number in the line is the root.

Illustratio

| – | – |		
5 4 7 5 6		
4	Double the root 2=4	
4) 1 4 (3	Setting down double this quotient,	
1 2	*i.e.*, 3 × 2=6 in the line we get	46
2 7		
9		
46) 1 8 5 (4	Setting down double this quotient,	
1 8 4	*i.e.*, 4 × 2=8 in the line we get	468
1 6		
1 6		

The root=half the number in the line, *i.e.*, $\frac{468}{2}=234$.

Kamalākar (1658) gives the rule thus :*

अंकं यावदिहाद्यांकादूर्ध्वं तिर्यक्स्थ रेखया ।
संज्ञा स्थानाङ्कानां च विषमाख्य सम क्रमात् ॥
त्यक्त्वान्त्याद्विषमाद्वर्गं द्विघ्न तन्मूल हृत समः ।
लब्ध वर्गं च विषमाद्याच्छोध्यं पुनः पुनः ॥
क्रियैवं सर्वं मूलांकं यावत्तत्र पदानि च ।
अन्त्यस्थानोत्क्रमेणैवं मूलं स्वीयमुदाहृतम् ॥

i.e., By means of vertical and horizontal lines placed on the digits (of the given number), from the first to the last, the digits standing in the several places are designated as odd and even respectively. Subtract the square number from the last odd place, and divide the even place by twice the root; then subtract from the next odd place the square of this quotient; repeat this procedure again and again till all the digits of the root (are obtained). The roots of the squares (that have been subtracted) placed from the last place in the reverse order are termed the root.†

It is thus clear that all the Indian mathematicians from Āryabhaṭa (b. 476 A. D.) to Kamalākar (1658) have given the same method of extracting square roots. We shall now show that this method,

* *Siddhānta-tattva-viveka*, ed. by S. Dvivedi, p. 46.

† See worked out example, pp. 125-26. The roots of the squares subtracted are 2, 3, and 4 : thus the root is at first 2, then 23 and then 234.

which we can now call the *Indian Method*, is the same as the present method.

Present method the same as Āryabhaṭa's method.

For comparison, the working of Āryabhaṭa's method of extracting square roots is given below side by side with the present method.

Āryabhaṭa's method.

```
              |  -  |  -  |
              5  4  7  5  6
   2² =       4    ...              (1)  root = 2

 2×2 = 4 ) 14 ( 3......(a)      }
           12                   }        Placing quo-
           —                    }   (2)  tient at the
           27                   }        next  place
   3² =     9 .........(b)      }        gives 23.

23×2 = 46 ) 185 ( 4...(a)       }
            184                 }        Pacing quo-
            ———                 }   (3)  tient at the
             16                 }        next  place
   4² =      16.........(b)     }        gives 234.
```

Present method.

```
      | 5 4 7  5 6 ( 234
      | 4          .........(1)
 43   | 147
      | 129        .........(2)
 464  | 1856
      | 1856       ,,, .....(3)
```

It will be seen from the above illustration that the present method of extracting square roots is the same as Āryabhaṭa's method with the only difference that the processes indicated as (2) and (3) in the working of the present method were each carried out by Āryabhaṭa in two distinct steps marked (*a*) and (*b*) in the above illustration.

It is now established that a method of extracting square roots which is in all essentials the same as the present method was known to Āryabhaṭa as early as 499 A. D. From the terse and abridged style adopted by him in giving the rule it seems that the method must have been well known among the mathematicians of his time. In fact there is nothing to show that Āryabhaṭa was its inventor. He was only twenty-three years of age when he wrote the *Āryabhaṭīya*. Moreover he does not lay claim to originality for the contents of his book, and definitely states that he has written what he obtained from his *guru* (teacher), and from the vast store of knowledge already known. The invention of the method, therefore, may be said to have been made about the beginning of the fifth century of our era if not earlier.

Along with the Hindu Numerals, the Indian method of extracting square roots seems to have been communicated to the Arabs about the middle of the eighth century, for it occurs in the writings of all Arab mathematicians who wrote on the Hindu numerals in precisely the same form as given by the Indian mathematicians. Thence it was

communicated to Europe, and occurs in similar form in the writings of Peurbach* (1423-1461), Chuquet (1484), La Roche (1520), Gemma Frisius (1540), Cataneo (1546), and others.

II.

The extraction of Cube Root

It will now be shown that the method of extracting cube roots given by the ancient Indian mathematicians is the same in all essentials as the present method. The first Indian mathematician to give the method was Āryabhaṭa. The method as given in the *Āryabhaṭīya* runs as follows :

अघनाद् भजेद्द्वितीयात् त्रिघनस्य मूलवर्गेण ।
वर्गस्त्रिपूर्वे गुणितश्शोध्यः प्रथमाद् घनश्च घनात् ॥†

Kaye‡ has given an incorrect translation of the above rule, and has made the mistaken assertion that the method is algebraical in character and applies to all possible notations. His translation runs as follows :

" Multiply the square of the root of the cubic quantity by three, and divide the second non-cubic part by the product. Multiply the square of this by three times the preceding and subtract the product from the first non-cubic. Then the cube is to be subtracted from the cube. "

The above translation as it stands is perfectly meaningless. It is difficult to guess what Kaye really means by *the first non-cubic part* and *the second non-cubic part*, and subtracting the *cube from the cube.* The learned author of these translations has not illustrated the rule as given in his translation by an example, perhaps for the simple reason that he did not know what to make of it. Had Kaye possessed a moderate knowledge of Sanskrit, and a fair amount of acquaintance with the writings of Indian mathematicians, he would not have given such an absurd translation, especially in view of the fact that the published commentary of Parameśvar explains the method very lucidly and in very simple Sanskrit.§

* Truelein, " Das Rechnen in 16. Jahrhundert, " in *Abhandl. Math. Wissenschaft* (1877) ; *Triparty* in Boncompagni's *Bulletino di bibliografia e di storia delle scienze math. e fisiche*, XIII, p. 695.

† *Āryabhaṭīya*, Gaṇita-pāda, 5.

‡ *Journ. Asiatic Soc. Bengal*, Vol IV. (*Āryabhaṭa*).

§ The commentator Sūryadeva has explained the method with the help of an illustrative example.

The correct literal rendering of the rule ought to run as follows:

After having substracted the cube (of the quotient) from the *ghana** place, divide the second *aghana* place by thrice the square of the root, and subtract the square of the quotient multiplited by thrice the previous root from the next (*aghana* place); the quotient placed at the next place gives the root.†

The digits of the number whose cube root is to be found are, beginning with the units place, marked off into periods of three each of which the first is called the '*ghana*' the second the '*first aghana*' and the third the '*second aghana*'

Illustration

	1 9 5 3 1 2 5		
Subtract cube.‡	1		
Divide by thrice square of root, *i.e.*, 3×1^2 =	3) 9 (2 §	..... (*a*)	
Subtract square of quotient multiplied by thrice the previous root, *i.e.*, $2^2 \times 3 \times 1$ =	6		Root = 1
	3 5		Placing quotient after the root gives the root 12.
	1 2	..... (*b*)	
Subtract cube of quotient, *i.e.*, 2^3 =	2 3 3		
	8	...... (*c*)	
Divide by thrice square of root, *i.e.*, 3×12^2 =	432)2 2 5 1 (5	...... (*a*)	
	2 1 6 0		
Subtract square of quotient multiplied by thrice the previous root, *i.e.*, $5^2 \times 3 \times 12$ =	9 1 2		Placing quotient after the root gives the root 125.
	9 0 0	...... (*b*)	
Subtract cube of the quotient, *i.e.*, 5^3 =	1 2 5		
	1 2 5	...... (*c*)	

∴ Cube root = 125.

* प्रथम स्थानं घन संज्ञं द्वितीय त्रितीये अघनं संज्ञे चतुर्थे घन संज्ञं, पंचम षष्ठे अघन संज्ञे एवं अन्यापि (Parameśvar's Commentary). The first place is called '*ghana*,' the second and third are '*aghana*', the fourth '*ghana*', the fifth and the sixth '*aghana*' and so on.

† This portion is common to this and the previous couplet, see page 124.

‡ अन्त्याद् घनस्थानाद्यथा लब्धं घनं विशोधयेत् (Com.). From the last '*ghana*' place subtract the cube number.

§ The division is to be carried to such an extent that the processes (*b*) and (*c*) can be carried out.

The working of the above example according to our present method would run thus :

$$
\begin{array}{lll r|l}
 & & & & \dot{1}\ 9\ \dot{5}\ \dot{3}\ 1\ 2\ \dot{5} \quad (125 \\
 & & & & 1 \\
\cline{5-5}
\multirow{3}{*}{(1)} & \left\{\begin{array}{l}(a')\ 1^2\times 300 \\ (b')\ 1\times 30\times 2 \\ (c')\qquad\quad 2^3\end{array}\right. & \begin{array}{l} = \\ = \\ = \end{array} & \begin{array}{r} 300 \\ 60 \\ 4 \end{array} & 9\ 5\ 3 \\
\cline{4-4}
 & & & 364 & 7\ 2\ 8 \\
\cline{5-5}
\multirow{3}{*}{(2)} & \left\{\begin{array}{l}(a')\ 12^2\times 300 \\ (b')\ 12\times 30\times 5 \\ (c')\qquad\quad 5^3\end{array}\right. & \begin{array}{l} = \\ = \\ = \end{array} & \begin{array}{r} 43200 \\ 1800 \\ 25 \end{array} & 2\ 2\ 5\ 1\ 2\ 5 \\
\cline{4-4}
 & & & 45025 & 2\ 2\ 5\ 1\ 2\ 5 \\
\cline{5-5}
 & & & & \quad \times
\end{array}
$$

It will be seen from the above illustration that the present method of extracting cube roots is simply a contraction of Āryabhaṭa's method. The steps marked (1) and (2) in the above illustration were each carried out by Āryabhaṭa in three steps marked (a), (b) and (c). The digits of the root in both processes are obtained by trial division. In fact the two methods are in all essentials the same.

All other Indian mathematicians who followed Āryabhaṭa have given the same rule. Brahmagupta (628 A. D.) gives it thus :

छेदो घनाद्द्वितीयाद्घनमूलकृतिस्त्रिसङ्गुणाप्त कृतिः ।
शोध्या त्रिपूर्वगुणिता प्रथमाद्घनतो घनो मूलम् ॥

The divisor for the second *aghana* place is thrice the square of the cube root. The square of the quotient multiplied by three and the preceding must be subtracted from the next (*aghana*), and the cube (of the quotient) from the *ghana* place : the root is found.*

* *Brāhmasphuṭa Siddhānta*, ed. by S. Dvivedi, p. 175 : for the translation see Colebrooke's *Algebra and Arithmetic from the Sanscrit of Brahmagupta and Bhāscara*, p. 280.

Srīdhara (*c.* 750) gives the method* thus :

घनपदमघनपदद्वे घनपदतोऽपास्य घनमतो मूलम् ।
संयोज्य तृतीय पदस्याधस्तदनष्ट वर्गेण ॥
एकस्थानेन तथा शेषे त्रिगुणेन संभजेत् तस्मात् ।
लब्धं निवेश्य पंक्त्यां तद्वर्गं त्रिगुणमन्त्य हतम् ॥
जह्यादुपरिमराशेः प्राग्वद् घनमादिमस्य च स्वपदात् ।
भूयस्तु तृतीय पदस्याध इत्यादि विधि मूलम् ॥

* The following is the orthodox method of carrying out the operation on a *pāṭi* :—

The number whose cube root is to be found is written on the *pāṭi* thus :

```
| - - | - - |
1 9 5 3 1 2 5
```

From the last *ghana* place the highest possible cube is subtracted. Here 1 being subtracted from 1 gives nothing. 1 is rubbed out, and the root of the cube, *i.e.*, 1 is now written down permanently in a separate line, and also beneath the third figure of the next period thus

```
- - | - - |
9 5 3 1 2 5        separate line
1                        1
```

Thrice the square of 1 is 3. Divide by this the next place, *i.e.*, 9. (Here we have to take 2 as the quotient, for if we divide out completely the rest of the procedure can not be carried out.) The quotient is 2, and the remainder 3. 9 is rubbed out and the remainder 3 written in its place. The number on the *pāṭi* becomes

```
- | - - |
3 5 3 1 2 5
```

The quotient 2 being placed in the line, the figures in the separate line become 12.

The square of the quotient is 4. This multiplied by 3 and the previous root, *i.e.*, $4 \times 3 \times 1 = 12$. Subtracting 12 from the figures upto the next place, *i.e.*, 35 gives 23. 35 is rubbed out and 23 written in its place. The number on the *pāṭi* now becomes

```
| - - |
2 3 3 1 2 5
```

Translation: (Divide into periods of) one *ghana* and two *aghana* places. From the (last) *ghana* place subtract the (greatest possible) cube, placing its root below the third place (*i e.*, the second *aghana* of the next period) and also permanently (in a separate line) and divide by thrice the square of this one figured root the next place (*i.e.*, the second *aghana*), and having placed the quotient in the line, subtract its square multiplied three times the preceding (root) from the number represented by the figures upto the next place, and from the first place (of the period) (*i.e.*, its own place) its cube ; again (repeating) the same procedure as before—below the third place etc.—gives the root.

Mahāvīrāchārya* (*c.* 850) has expressed the method in two ways. The first is

अन्त्यघनादपहृत घन मूलकृति त्रिकृति भजिते भाज्ये ।
प्राक्त्रि कृताद्यस्य कृतिर्गुणा शोध्ये घनेऽथ घनं ॥

Translation: From the last *ghana* place subtract the (greatest possible) cube ; then divide the *bhājya* place by three times the square of

The cube of the quotient is now subtracted from the first place of the period, *i.e.*, the *ghana* place. Subtracting 2^3, *i. e.*, 8 from 233 gives 225. 233 is rubbed out and 225 written in its place. The number on the *pāṭi* now becomes

2 2 5 1 2 5

Placing the root so far obtained (*i. e.*, the number in the line), *i. e.*, 12 below the third place of the next period and dividing by thrice its square (*i. e.*, $3 \times 12^2 = 432$) the figures upto the next place, *i.e*, 2251, we get 5 as quotient and 91 as remainder. 2251 is rubbed out and the remainder 91 written in its place. The number on the *pāṭi* now becomes

9 1 2 5

The quotient being placed in the line, the figures in the line become 125.

The square of the quotient multiplied by thrice the previous root (*i. e.*, $5^2 \times 3 \times 12 = 900$) is subtracted from the next place, *i. e.*, 912. This gives 12 as remainder. 912 is rubbed out and 12 written in its place. The number on the *pāṭi* now becomes

1 2 5

The cube of the last quotient, *i. e.*, 5^3 is subtracted from the figures upto the first place, *i. e.*, the *ghana* place. This leaves no remainder and the process ends.

The number in the line, *i. e.*, 125 is the required root.

* *Gaṇitasāra Saṃgrah*, ed. by M. Rangāchārya, p. 16.

the root ; then subtract from the *śodhya* place the square of the above quotient (and so on).*

The second is

घनमेकं द्वे अघने घनपदकृत्या भजेत्त्रिगुणयास्य घनतः ।
पूर्वं त्रिगुणाप्तकृतिघ्नात्त्यजेत्तस्य घनश्च पूर्ववल्लब्धैः पदैः ॥

Translation: The first place is (called) *ghana*, the next two are *aghana* (and so on). Divide the *second aghana* by three times the square of the root. From the (next) *aghana* subtract the square of the quotient multiplied by three times the previously obtained root, and then subtract the cube of the quotient (from the next *ghana* place). With the help of the root figures (so) obtained (and taken into position, the procedure is) as before.†

Āryabhaṭa II (*c.* 950) gives the rule thus :‡

घनभाज्यशोध्यसंज्ञानि पदानि घनं त्यजेत् स्वपदात् ।
मूलं भाज्यपदाधो निधाय तदधस्ट वर्गेण ॥
त्रिगुणेन भजेत् स्वपदाल्लब्धं विनिवेश्य पंक्तौ तत् ।
वर्गं त्रिपूर्वगुणितं शोध्यात् घनं च घन पदत: ॥
तन्मूलं भाज्याधो निधाय कार्यो विधिः प्राग्वत् ।

Translation : *Ghana*, *bhājya* and *śodhya* are the denominations of the digits (according to their position). Subtract the cube from its own place (*i. e.*, the *ghana* place) writing-down the root permanently (in a separate line) and also below the *bhājya* place, and divide by thrice the square of this (root) its own place (*i. e.*. the *bhājya* place), and having placed the quotient in the line, subtract its square multiplied by three

* *Ibid*, translation, p. 18.

The only difference between this mode of expressing the rule and that of Āryabhaṭa is that the *second aghana place* is called by Mahāvīrāchārya the *bhājya place* (meaning the place on which the operation of division is to be performed), while the *first aghana place* is called the *śodhya place* (meaning the place from which subtraction is to be made).

† *Ibid*, p. 16.

‡ *Mahāsiddhānta*, ed. by S. Dvivedī, p. 145.

times the preceding (root) from the *śodhya* place. Placing the root so far obtained below the *bhājya* place the process continues as before.

Bhāskarāchārya (*b*. 1114) gives the rule thus :

आद्यं घनस्थानमथाघने द्वे, पुनस्तथान्त्याद् घनतो विशोध्य ।
घनं पृथक्स्थं पदमस्य कृत्या, त्रिघ्न्या तदाद्यं विभजेत फलन्तु ॥
पंक्त्यां न्यसेत् तत् कृतिमन्त्य निघ्नी, त्रिघ्नीं त्यजेत्तत्प्रथमात् फलस्य ।
घनं तदाद्याद् घनमूलमेवं, पंक्तिर्भवेदेवमतः पुनश्च ॥

Translation: The first digit is a *ghana's* place, and the two next *aghana* ; and again the rest in like manner. From the last *ghana* place take out the nearest cube, and set down its root apart. By thrice the square of that root divide the next (or *aghana*) place of figures, and note the quotient in the line (in which the previous root stands). Subtract its square multiplied by thrice the last (root) from the next (place) and its cube from the succeeding one. Thus the line (in which the result is reserved) is the root (figures so far obtained). The operation is repeated as necessary).*

We have now shown that a method of extracting cube roots which is in all essentials the same as the present method has been given by the ancient Indian mathematicians from Āryabhaṭa (499 A. D.) onwards. It is also clear from the illustrations that have been given that the methods of extracting square and cube roots given by the Indian mathematicians are particularly meant to apply to the decimal place value notation, and are not algebrical in character. Kaye's assertion that the methods are algebrical in character is absolutely wrong. It appears that Kaye in his zeal to uphold the Arabic origin of the Hindu decimal place value notation has deliberately tried to misrepresent the Indian mathematicians by giving wrong and absurd translations of their rules for the extraction of square and cube roots, for if these rules are correctly translated, they show that the decimal place value notation was commonly known in India as early as 499 A.D., *i. e.*, long before the birth of Islām and the Arab nation.

It has already been pointed out that the ancient Greeks did not possess methods for the extraction of square and cube roots in any way resembling our present methods, so that, on the evidence at hand, one

* See Colebrooke, *Algebra and Arithmetic from the Sanscrit of Brahmagupta and Bhāscara*, p. 280.

is forced to conclude that *we are indebted to the ancient Indian mathematicians for our present methods of extracting square and cube roots, and that the methods were well known among the Indian mathematicians about* 499 A.D., *and there is every reason to believe that they were invented much earlier.*

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3

On the strain in a rotating elliptic cylinder

BY

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When an elastic cylinder is made to rotate rapidly about its axis a state of strain is set up which may be regarded as plane strain if the cylinder be sufficiently long and suitable tractions are applied at the plane ends to prevent any change in length. When the plane ends are free, the length of the cylinder will undergo a change under the influence of rotation and in this case the state of strain may be regarded as one of plane strain attended with uniform longitudinal extension. Making these assumptions the problem of a rotating circular cylinder has been solved * but it appears that the problem of a cylinder whose cross section is any other than a circle has not yet been solved. In fact the equations are too complicated to admit of easy solution. In the present paper the case of the elliptic cylinder has been investigated and a solution obtained which represents the state of stress and strain fairly accurately when the eccentricity of the cross section of the cylinder is small enough.

2. Let the axis of the cylinder be taken as the axis of z and $z=\pm l$ be the equations of the plane ends Let us use the orthogonal co-ordinates α, β, γ given by

$$x=c \cosh \alpha \cos \beta,$$
$$y=c \sinh \alpha \sin \beta,$$
$$z=\gamma,$$

so that the surfaces α=constant represent a family of cylinders having a common axis, *viz.*, the axis of z and their cross-sections are a family of confocal ellipses, the distance between the foci being $2c$.

* See Love's *Theory of Elasticity*, 2nd edition, Art. 102.

If h_1, h_2, and h_3 denote the modulæ of transformation, we have

$$h_1=\left\{\left(\frac{\partial x}{\partial \alpha}\right)^2+\left(\frac{\partial y}{\partial \alpha}\right)^2+\left(\frac{\partial z}{\partial \alpha}\right)^2\right\}^{-\frac{1}{2}}$$

$$=\frac{\sqrt{2}}{c}(\cosh 2\alpha-\cos 2\beta)^{-\frac{1}{2}};$$

$$h_2=\left\{\left(\frac{\partial x}{\partial \beta}\right)^2+\left(\frac{\partial y}{\partial \beta}\right)^2+\left(\frac{\partial z}{\partial \beta}\right)^2\right\}^{-\frac{1}{2}}$$

$$=\frac{\sqrt{2}}{c}(\cosh 2\alpha-\cos 2\beta)^{-\frac{1}{2}};$$

$$h_3=\left\{\left(\frac{\partial x}{\partial \gamma}\right)^2+\left(\frac{\partial y}{\partial \gamma}\right)^2+\left(\frac{\partial z}{\partial \gamma}\right)^2\right\}^{-\frac{1}{2}}$$

$$=1.$$

Thus $h_1=h_2=h$ (say)

$$=\frac{\sqrt{2}}{c}(\cosh 2\alpha-\cos 2\beta)^{-\frac{1}{2}} \qquad \ldots \quad (1)$$

3. Adopting Love's notation* and treating the problem as one of plain strain together with a uniform extension e along the z-axis, we have the following expressions for the components of strain :

$$\left.\begin{aligned}
e_{\alpha\alpha}&=h\frac{\partial u_\alpha}{\partial \alpha}+h^2 u_\beta\frac{\partial}{\partial \beta}\left(\frac{1}{h}\right),\\
e_{\beta\beta}&=h\frac{\partial u_\beta}{\partial \beta}+h^2 u_\alpha\frac{\partial}{\partial \alpha}\left(\frac{1}{h}\right),\\
e_{\gamma\gamma}&=e,\\
e_{\beta\gamma}&=0,\\
e_{\gamma\alpha}&=0,\\
e_{\alpha\beta}&=\frac{\partial}{\partial \alpha}(h\,u_\beta)+\frac{\partial}{\partial \beta}(h\,u_\alpha),
\end{aligned}\right\} \qquad \ldots \quad (2)$$

* *Theory of Elasticity*, 2nd edition, p. 54.

u_α, u_β and u_γ $(=e\gamma)$ being the components of displacement normal to the surfaces α, β, and γ respectively and u_α, u_β being independent of γ.

4. If f_α, f_β, f_γ denote the components of acceleration along the normals to the surfaces α, β, γ respectively and ρ denote the density of the material of the cylinder (supposed homogeneous and istropic) and Δ, ϖ_α, ϖ_β, ϖ_γ, λ, μ have their usual meanings, the equations of motion are :

$$(\lambda+2\mu)h_1\frac{\partial\Delta}{\partial\alpha}-2\mu h_2h_3\frac{\partial}{\partial\beta}\left(\frac{\varpi_\gamma}{h_3}\right)+2\mu h_2h_3\frac{\partial}{\partial\gamma}\left(\frac{\varpi_\beta}{h_2}\right)=\rho f_\alpha,$$

$$(\lambda+2\mu)h_2\frac{\partial\Delta}{\partial\beta}-2\mu h_3h_1\frac{\partial}{\partial\gamma}\left(\frac{\varpi_\alpha}{h_1}\right)+2\mu h_3h_1\frac{\partial}{\partial\alpha}\left(\frac{\varpi_\gamma}{h_3}\right)=\rho f_\beta,$$

$$(\lambda+2\mu)h_3\frac{\partial\Delta}{\partial\gamma}-2\mu h_1h_2\frac{\partial}{\partial\alpha}\left(\frac{\varpi_\beta}{h_2}\right)+2\mu h_1h_2\frac{\partial}{\partial\beta}\left(\frac{\varpi_\alpha}{h_1}\right)=\rho f_\gamma.$$

It can be easily verified that in the present problem Δ is independent of γ while $\varpi_\alpha=\varpi_\beta=0$. Also $f_\gamma=0$, $h_3=1$ and $h_1=h_2=h$. The third equation is therefore identically satisfied and the first two equations reduce to

$$(\lambda+2\mu)h\frac{\partial\Delta}{\partial\alpha}-2\mu h\frac{\partial}{\partial\beta}(\varpi_\gamma)=-\rho\omega^2\left(xh\frac{\partial x}{\partial\alpha}+yh\frac{\partial y}{\partial\alpha}\right)$$

and $$(\lambda+2\mu)h\frac{\partial\Delta}{\partial\beta}-2\mu h\frac{\partial}{\partial\alpha}(\varpi_\gamma)=-\rho\omega^2\left(yh\frac{\partial x}{\partial\alpha}-xh\frac{\partial y}{\partial\alpha}\right)$$

i.e., to

$$(\lambda+2\mu)\frac{\partial\Delta}{\partial\alpha}-2\mu\frac{\partial}{\partial\beta}(\varpi_\gamma)=-\frac{1}{2}\rho\omega^2c^2\sinh 2\alpha \qquad \dots \quad (3)$$

and $$(\lambda+2\mu)\frac{\partial\Delta}{\partial\beta}-2\mu\frac{\partial}{\partial\alpha}(\varpi_\gamma)=\frac{1}{2}\rho\omega^2c^2\sin 2\beta \qquad \dots \quad (4)$$

where ω denotes the uniform angular velocity of the cylinder.

If $\alpha=\alpha_0$ be the equation of the curved surface of the cylinder and $\gamma=\pm l$ the equations of the plane ends, the boundary conditions are :

$$\widehat{\alpha\alpha}=\widehat{\alpha\beta}=\widehat{\gamma\alpha}=0, \quad \text{when } \alpha=\alpha_0$$

and if the plane ends be free,

$$\widehat{\gamma\gamma}=\widehat{\beta\gamma}=\widehat{\gamma\alpha}=0, \quad \text{when } \gamma=\pm l.$$

5. In the present problem

$$\widehat{\beta\gamma}=\widehat{\gamma\alpha}=0, \quad \text{everywhere},$$

since $$e_{\beta\gamma}=e_{\gamma\alpha}=0.$$

To obtain the particular solution of the equations (3) and (4) we assume in the first instance a solution of the type

$$\left.\begin{aligned} u_\alpha&=U_\alpha/h \\ u_\beta&=U_\beta/h \end{aligned}\right\} \qquad \ldots \qquad (5)$$

where U_α involves only α and U_β only β. It is clear from the equations (2) that under this assumption $e_{\widehat{\alpha\beta}}=0$ and accordingly

$$\widehat{\alpha\beta}=0$$

everywhere.

We also find

$$\Delta = e_{\alpha\alpha}+e_{\beta\beta}+e_{\gamma\gamma}$$

$$= \frac{\partial U_\alpha}{\partial \alpha}+\frac{\partial U_\beta}{\partial \beta}+h^2c^2\left(U_\alpha \sinh 2\alpha+U_\beta \sin 2\beta\right) \qquad (6)$$

and $$2\varpi_\gamma = h^2\left\{\frac{\partial}{\partial \alpha}\left(\frac{u_\beta}{h}\right)-\frac{\partial}{\partial \beta}\left(\frac{u_\alpha}{h}\right)\right\}$$

$$= c^2h^4\left(U_\beta \sinh 2\alpha - U_\alpha \sin 2\beta\right) \qquad (7)$$

Substituting these values in the equations (3) and (4) and making use of the relation (1) we have the following two equations after some simplifications :

$$(\lambda+2\mu)\left[\frac{\partial^2 U_\alpha}{\partial\alpha^2}+c^2h^2\left(\frac{\partial U_\alpha}{\partial\alpha}\sinh 2\alpha+2U_\alpha\cosh 2\alpha\right)\right.$$

$$\left.-c^4h^4\sinh 2\alpha\left(U_\alpha\sinh 2\alpha+U_\beta\sin 2\beta\right)\right]$$

$$-\mu\left[c^2h^2\left(\frac{\partial U_\beta}{\partial\beta}\sinh 2\alpha-2U_\alpha\cos 2\beta\right)\right.$$

$$\left.-c^4h^4\sin 2\beta\left(U_\beta\sinh 2\alpha-U_\alpha\sin 2\beta\right)\right]$$

$$=-\frac{1}{2}\rho\omega^2c^2\sinh 2\alpha. \qquad \ldots \qquad (8)$$

$$(\lambda+2\mu)\left[\frac{\partial^2 U_\beta}{\partial\beta^2}+c^2h^2\left(\frac{\partial U_\beta}{\partial\beta}\sin 2\beta+2U_\beta\cos 2\beta\right)\right.$$

$$\left.-c^4h^4\sin 2\beta\left(U_\alpha\sinh 2\alpha+U_\beta\sin 2\beta\right)\right]$$

$$+\mu\left[c^2h^2\left(2U_\beta\cosh 2\alpha-\frac{\partial U_\alpha}{\partial\alpha}\sin 2\beta\right)\right.$$

$$\left.-c^4h^4\sinh 2\alpha\left(U_\beta\sinh 2\alpha-U_\alpha\sin 2\beta\right)\right]$$

$$=\frac{1}{2}\rho\omega^2c^2\sin 2\beta \qquad \ldots \qquad (9)$$

It is easily verified that the above two equations are both satisfied by

$$U_\alpha = A\sinh 2\alpha$$

and $$U_\beta = A\sin 2\beta,$$

where $\qquad A = -\dfrac{\rho\,\omega^2 c^2}{16\,(\lambda+2\mu)}$

Thus a particular solution of the equations of motion which makes $\widehat{\alpha\beta}=\widehat{\beta\gamma}=\widehat{\gamma\alpha}=0$ everywhere, is given by

$$\left.\begin{aligned} u_\alpha &= -\frac{\rho\omega^2 c^2}{16(\lambda+2\mu)}\cdot\frac{\sinh 2\alpha}{h} \\ u_\beta &= -\frac{\rho\omega^2 c^2}{16(\lambda+2\mu)}\cdot\frac{\sin 2\beta}{h} \end{aligned}\right\} \tag{10}$$

We may superpose on this solution any solution of the equations (2) and (3) with their righthand members equated to zero. Such a solution which in addition makes the stress-components $\widehat{\alpha\beta}$, $\widehat{\beta\gamma}$ and $\widehat{\gamma\alpha}$ vanish everywhere is the solution given by

$$\left.\begin{aligned} u_\alpha &= B\,h\sinh 2\alpha, \\ \text{and}\qquad u_\beta &= -\,B\,h\sin 2\beta, \end{aligned}\right\} \tag{11}$$

where B is an arbitrary constant.

Thus combining the two solutions (10) and (11) we find the following results :—

$$e_{\alpha\alpha} = \frac{2\,B}{c^2} - \frac{\rho\omega^2 c^2}{16(\lambda+2\mu)}\quad(3\cosh 2\alpha + \cos 2\beta)$$

$$e_{\beta\beta} = \frac{2\,B}{c^2} - \frac{\rho\omega^2 c^2}{16(\lambda+2\mu)}\quad(\cosh 2\alpha + 3\cos 2\beta)$$

$$e_{\gamma\gamma} = e$$

$$e_{\beta\gamma} = e_{\gamma\alpha} = e_{\alpha\beta} = 0$$

$$\Delta = e_{\alpha\alpha} + e_{\beta\beta} + e_{\gamma\gamma}$$

$$= \frac{4\,B}{c^2} + e - \frac{\rho^2\omega^2 c^2}{4(\lambda+2\mu)}\ (\cosh\ 2\alpha + \cos\ 2\beta)$$

$$\widehat{\alpha\alpha} = \lambda\Delta + 2\mu e_{\alpha\alpha}$$

$$= \frac{4\,B}{c^2}(\lambda+\mu) + \lambda e$$

$$-\frac{1}{8}\frac{\rho\omega^2 c^2}{\lambda+2\mu}\left[(2\lambda+3\mu)\cosh\ 2\alpha + (2\lambda+\mu)\cos 2\beta\right]$$

$$\widehat{\beta\beta} = \lambda\Delta + 2\mu e_{\beta\beta}$$

$$= \frac{4B}{c^2}(\lambda+\mu) + \lambda e$$

$$-\frac{1}{8}\frac{\rho\omega^2 c^2}{\lambda+2\mu}\left[(2\lambda+\mu)\cosh\ 2\alpha + (2\lambda+3\mu)\cos\ 2\beta\right]$$

$$\widehat{\gamma\gamma} = \lambda\Delta + 2\mu e_{\gamma\gamma}$$

$$= \frac{4B\lambda}{c^2} + (\lambda+2\mu)e - \frac{\lambda\rho\omega^2 c^2}{4(\lambda+2\mu)}\ (\cosh\ 2\alpha + \cos\ 2\beta)$$

$$\widehat{\alpha\beta} = \widehat{\beta\gamma} = \widehat{\gamma\alpha} = 0$$

The boundary condition $\widehat{\alpha\alpha}=0$ when $\alpha=\alpha_0$ gives

$$\frac{4\,B}{c^2}(\lambda+\mu) + \lambda e - \frac{\rho\omega^2 c^2}{8(\lambda+2\mu)}\left[(2\lambda+3\mu)\ \cosh\ 2\alpha_0 - (2\lambda+\mu)\cos\ 2\beta\right] = 0$$

This equation cannot obviously be satisfied by any constant values of B and e but if the eccentricity of the cross-section of the cylinder be small, we may neglect the term involving cos 2β and then to a first approximation we have

$$\frac{4B}{c^2}(\lambda+\mu)+\lambda e - \frac{1}{8}\rho\omega^2 c^2 \frac{2\lambda+3\mu}{\lambda+2\mu}\cosh 2\alpha_0=0 \qquad \dots \quad (12)$$

It may be noticed that although the normal surface traction $\widehat{\alpha\alpha}$ does not vanish, both the force-component as well as the couple-component of the resultant surface traction vanishes as can be easily verified by working out the three integrals

$$X=\int_0^{2\pi}\widehat{\alpha\alpha}\frac{\partial x}{\partial \alpha}\,d\beta$$

$$Y=\int_0^{2\pi}\widehat{\alpha\alpha}\,.\,\frac{\partial y}{\partial \alpha}\,d\beta$$

$$L=\int_0^{2\pi}\widehat{\alpha\alpha}\,.\,\frac{p}{h}\,d\beta$$

where p is the shortest distance between the normal at any point of the surface and the axis of the cylinder. It will be found that $X=Y=L=0$.

The boundary condition $\widehat{\gamma\gamma}=0$ at $\gamma=\pm l$ (*vide* Art. 4) is not satisfied but the value of the longitudinal extension may be properly adjusted so as to make the resultant tractions on the plane ends vanish. This requires

$$\int_0^{2\pi}d\beta\int_0^{\alpha_0}\frac{\widehat{\gamma\gamma}}{h^2}\,d\alpha=0.$$

Substituting the values of $\widehat{\gamma\gamma}$ and h^2 and effecting the integration, we get the equation

$$\frac{4\,B\lambda}{c^2}+(\lambda+2\mu)e-\frac{1}{8}\frac{\lambda\rho\omega^2c^2}{\lambda+2\mu}\cosh 2\alpha_0=0 \qquad \dots \quad (13)$$

Solving the two equations (12) and (13) we find

$$\left.\begin{aligned} B &= \frac{1}{32}\rho\omega^2c^4.\ \frac{\lambda^2+7\lambda\mu+5\mu^2}{\mu(\lambda+2\mu)(3\lambda+2\mu)}\cosh 2\alpha_0 \\ e &= -\frac{1}{8}\rho\omega^2c^2\,.\frac{\lambda}{\mu(3\lambda+2\mu)}\cosh 2\alpha_0 \end{aligned}\right\} \qquad \dots \quad (14)$$

Thus the solution of the problem to a first approximation is given by

$$\left.\begin{aligned} u_\alpha &= \left(Bh-\frac{\rho\omega^2c^2}{16(\lambda+2\mu)}\cdot\frac{1}{h}\right)\sinh 2\alpha, \\ u_\beta &= -\left(Bh+\frac{\rho\omega^2c^2}{16(\lambda+2\mu)}\cdot\frac{1}{h}\right)\sin 2\beta, \\ \text{and} \qquad u_\gamma &= e\gamma, \end{aligned}\right\} \qquad \dots \quad (15)$$

where h is given by the equation (1) and B and e by the equations (14).

Instead of making the resultant longitudinal tension vanish we might suppose that the tension is adjusted so that the length is maintained constant. Then we should have

$$e=0,\ B=\frac{1}{32}\rho\omega^2c^4\ \frac{2\lambda+3\mu}{(\lambda+\mu)(\lambda+2\mu)}\cosh 2\alpha_0 \ \dots(16)$$

and the resultant tensions which it would be necessary to apply at the ends would be given by

$$Z=\int_0^{2\pi}d\beta\int_0^{\alpha_0}\frac{\widehat{\gamma\gamma}}{h^2}\,d\alpha$$

$$=\lambda\sinh 2\alpha_0\left[\frac{2B}{c^2}-\frac{1}{16}\cdot\frac{\rho\omega^2c^2}{\lambda+2\mu}\cosh 2\alpha_0\right]$$

$$=\frac{1}{32}\cdot\frac{\lambda}{\lambda+\mu}\,\rho\omega^2 c^2 \sinh 4\alpha_0 \qquad \ldots \quad (17)$$

Thus the equations (15) give an approximate solution of the problem, B and e being determined by the equations (14) when the cylinder rotates freely and by the equations (16) when tensions given by (17) are applied at the plane ends in order to maintain the constancy of the length.

The components of stress and strain can be easily calculated by substituting the proper values of B and e in the expressions for $e_{\alpha\alpha}$, ..., $\widehat{\alpha\alpha}$, ...already obtained.

The above solution represents the state of stress and strain fairly accurately at all points which are not too near the boundary.

It may be noted incidentally that the solution (11) corresponds to the problem of the equilibrium of an elliptic cylinder subjected to uniform normal pressure on its curved surface.

4

ON THE SUMMABILITY (C 1) OF THE FOURIER SERIES OF A FUNCTION AT A POINT WHERE THE FUNCTION HAS A DISCONTINUITY OF THE SECOND KIND

BY

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The object of the present paper is to formulate a definite answer to the question : Does Fejér's mean tend to a definite limit with increasing n at a point x_0, where the function $f(x)$ associated with the Fourier series has a discontinuity of the second kind so that

$$\operatorname*{Lim}_{z=0}\{f(x_0+2z)+f(x_0-2z)\}$$

is non-existent ?

For the sake of simplicity and fixity of ideas, I restrict my discussion of the above question to the case in which $f(x_0+2z)=\cos\psi(z)$, where $\psi(z)$ is monotone and tends to infinity with z tending to 0.

Leaving aside the trivial case of ψ being an odd function of z, it is proved that Fejér's mean tends to a definite limit or not, according as

$$\psi(z)' \succ \log\frac{1}{z^2}$$

or not.

It is believed that the above result has not been given by any previous writer.

1. The Fourier series of $f(x)$ is, as is well-known,

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt.$$

Thus Fejér's mean

$$S_n(x) \equiv \frac{s_0+s_1+\ldots+s_n}{n+1},$$

where

$$s_n = \tfrac{1}{2}a_0 + \sum_{r=1}^{r=n} (a_r \cos rx + b_r \sin rx),$$

is equal to

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi} \left\{\frac{\sin \frac{1}{2}n\,(t-x)}{\sin \frac{1}{2}\,(t-x)}\right\}^2 f(t)\, dt,$$

$$\text{i.e.} \quad \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} \left(\frac{\sin nz}{\sin z}\right)^2 \{f(x+2z)+f(x-2z)\}dz.$$

Therefore

$$S_n(x_0) = \frac{2}{n\pi}\int_0^{\frac{\pi}{2}} \left(\frac{\sin nz}{\sin z}\right)^2 \cos \psi\,(z)dz.$$

$$= \frac{2}{n\pi}\int_0^{a} \left(\frac{\sin nz}{\sin z}\right)^2 \cos \psi\,(z)dz + \frac{2}{n\pi}\int_a^{\frac{\pi}{2}} \left(\frac{\sin nz}{\sin z}\right)^2 \cos \psi\,(z)dz,$$

where a is an arbitrarily small quantity independent of n.

Now,

$$\left|\ \frac{2}{n\pi}\int_a^{\frac{\pi}{2}} \left(\frac{\sin nz}{\sin z}\right)^2 \cos \psi\,(z)dz\ \right|$$

$$< \frac{2}{n\pi}\int_a^{\frac{\pi}{2}} \frac{dz}{\sin^2 z}, \qquad \text{i.e.} \quad \frac{2}{\pi n}\cot a.$$

Therefore, with increasing n, $S_n(x_0)$ behaves as

$$\frac{2}{n\pi}\int_0^a \left(\frac{\sin nz}{\sin z}\right)^2 \cos\psi(z)dz,$$

$$i.e.\ \text{ as }\ \frac{2}{n\pi}\int_0^a \left(\frac{\sin nz}{z}\right)^2 \cos\psi(z)dz.$$

CASE : $\psi(z) \backsim \log\frac{1}{z^2}$

2. I proceed now to consider the various types of ψ's. First let

$$\psi(z) \backsim \log\frac{1}{z^2}.$$

The behaviour of $S_n(x_0)$ depends on

$$\frac{2}{n\pi}\int_0^a \left(\frac{\sin nz}{z}\right)^2 \cos\log\frac{1}{z^2}\,dz$$

$$i.e.\quad \frac{2}{\pi}\int_0^{na} \left(\frac{\sin\theta}{\theta}\right)^2 \cos\left(2\log\frac{n}{\theta}\right)d\theta$$

Hence, as n increases, $S_n(x_0)$ behaves as $R\cos(2\log n+\gamma)$, where

$$R\cos\gamma=\frac{2}{\pi}\int_0^\infty \left(\frac{\sin\theta}{\theta}\right)^2 \cos\left(2\log\frac{1}{\theta}\right)d\theta,$$

$$R\sin\gamma=\frac{2}{\pi}\int_0^\infty \left(\frac{\sin\theta}{\theta}\right)^2 \sin\left(2\log\frac{1}{\theta}\right)d\theta.$$

Therefore it is proved that $\underset{n=\infty}{\text{Lim}}\ S_n(x_0)$ is non-existent.

CASE : $\psi(z) \prec \log \frac{1}{z^2}$.

3. Choose a quantity ϵ which equals $\frac{C}{n}$, where C is a large quantity independent of n, so that

$$0 \leq \epsilon < a.$$

Now

$$\frac{2}{n\pi}\int_0^a \left(\frac{\sin nz}{z}\right)^2 \cos\psi(z)\,dz = I_1 + I_2, \quad \text{where}$$

$$I_1 = \frac{2}{n\pi}\int_0^{\epsilon} \left(\frac{\sin nz}{z}\right)^2 \cos\psi(z)\,dz,$$

$$I_2 = \frac{2}{n\pi}\int_0^{a} \left(\frac{\sin nz}{z}\right)^2 \cos\psi(z)\,dz.$$

But

$$I_1 = \frac{2}{\pi}\int_0^{C} \left(\frac{\sin\theta}{\theta}\right)^2 \cos\psi\left(\frac{\theta}{n}\right)d\theta$$

$$= \frac{2}{\pi}\int_0^{\epsilon_1} \left(\frac{\sin\theta}{\theta}\right)^2 \cos\psi\left(\frac{\theta}{n}\right)d\theta$$

$$+ \frac{2}{\pi}\int_{\epsilon_1}^{C} \left(\frac{\sin\theta}{\theta}\right)^2 \cos\psi\left(\frac{\theta}{n}\right)d\theta$$

where $0 < \epsilon_1 < C$ and ϵ_1 is independent of n and can be chosen to be as small as we please.

Therefore the first part of I_1 is negligible and I_1 behaves as

$$\frac{2}{\pi}\int_{\epsilon_1}^{C} \left(\frac{\sin\theta}{\theta}\right)^2 \cos\psi\left(\frac{\theta}{n}\right)d\theta,$$

$$i.e. \quad \text{as} \quad \frac{2}{\pi}\int_{\epsilon_1}^{C}\left(\frac{\sin\theta}{\theta}\right)^2 \cos(U-V)\,d\theta,$$

where

$$\psi\left(\frac{\theta}{n}\right) = U - V,$$

U standing for $\psi\left(\frac{\epsilon_1}{n}\right)$.

But

$$V = -\left\{\psi\left(\frac{\theta}{n}\right) - \psi\left(\frac{\epsilon_1}{n}\right)\right\}$$

$$= -\frac{\theta-\epsilon_1}{n}\cdot\psi'\left(\frac{\bar{\theta}}{n}\right)$$

where $\psi'(z)$ stands for $\frac{d\psi}{dz}$ and $\epsilon_1 < \bar{\theta} < \theta$. Therefore, since

$$\psi'(z) \prec \frac{1}{z},$$

$$V \prec \frac{\theta-\epsilon_1}{\bar{\theta}}, \quad i.e. \quad \prec 1.$$

Thus it is proved that I_1 behaves as

$$\frac{2}{\pi}\int_0^C\left(\frac{\sin\theta}{\theta}\right)^2 \cos U\,d\theta,$$

$$i.e. \quad \text{as} \cos\left\{\psi\left(\frac{\epsilon_1}{n}\right)\right\}.$$

4. Consider I_3 which is numerically less than

$$\frac{2}{n\pi}\int_{\epsilon}^{a}\frac{1}{z^2}\,dz,\quad i.e.\quad \frac{2}{n\pi}\left(\frac{1}{\epsilon}-\frac{1}{a}\right),$$

$$i.e.\quad \frac{2}{\pi}\left(\frac{1}{C}-\frac{1}{na}\right).$$

Then $|I_3| < \dfrac{2}{\pi C}$. But C can be chosen beforehand to be as large as we please. Therefore I_3 may be neglected in considering the behaviour of $S_n(x_0)$. Thus it is proved that $S_n(x_0)$ behaves as $\cos\left\{\psi\left(\frac{\epsilon_1}{n}\right)\right\}$ and therefore $\lim\limits_{n=\infty} S_n(x_0)$ is non-existent.

CASE: $\psi(z) \succ \log\frac{1}{z^2}$.

5. In order to consider this case, I shall apply the following result given by Prof. W. H. Young:* The derived series of the Fourier series of a function $F(x)$ converges (C 1) at a point x_0, if one of the derivates of $F(x)$ is known to be bounded in a neighbourhood of x_0 and

$$\lim_{h=0}\frac{F(x_0+h)-F(x_0-h)}{2h}$$

exists; the sum of the derived series is this limit.

Now let us take $F(x)$ to be

$$\int_0^x f(t)\,dt.$$

Then the Fourier series of $F(x)$ is

$$\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx),$$

* "On the convergence of the derived series of Fourier series" *Proceedings of the London Mathematical Society*, Series 2, Vol. 17, 1918, p. 230.

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\, dx$$

$$= -\frac{1}{n\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt = -\frac{b_n}{n},$$

and

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\, dx$$

$$= -\frac{(-1)^n}{n\pi}\int_{-\pi}^{\pi} f(t)\, dt + \frac{1}{n\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt$$

$$= -\frac{(-1)^n}{n}a_0 + \frac{a_n}{n}.$$

Therefore the Fourier series of $F(x)$ is

$$\tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} \frac{1}{n}(a_n \sin nx - b_n \cos nx)$$

$$-a_0 \sum_{1}^{\infty} \frac{(-1)^n}{n}\sin nx.$$

Thus the Fourier series of the function

$$G(x) \equiv F(x) - \frac{a_0 x}{2} - \frac{1}{2}A_0$$

is

$$\sum_{n=1}^{\infty} \frac{1}{n}(a_n \sin nx - b_n \cos nx)$$

and therefore the derived series of the Fourier series of $G(x)$ is

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

6. But it has been proved* by me that $F'(x)$ exists even at the point of discontinuity x_0 of $f(x)$ and equals 0. Therefore

$$\operatorname*{Lim}_{h=0} \frac{G(x_0+h)-G(x_0-h)}{2h}$$

exists and equals $-\frac{1}{2}a_0$. In the neighbourhood of x_0, $G'(x)$ being $f(x)-\frac{1}{2}a_0$ is bounded. Therefore the derived series of the Fourier series of $G(x)$ converges (C 1) at x_0 to $-\frac{1}{2}a_0$; hence the Fourier series of $f(x)$ converges (C 1) at x_0 to 0.

* See Vol. 16 of this *Bulletin*.

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1

On the division of the lemniscate into nine equal parts

BY

S. C. Mitra

The object of the present note is to find the values of

$$\operatorname{Sn}\frac{4K}{9}, \qquad \operatorname{Sn}\frac{8K}{9}, \text{ etc.},$$

when the modulus k is equal to i. As is well known, the problem of the division* of the arc of the lemniscate into nine equal parts is solved when the aforesaid values are known.

Let us suppose

$$\operatorname{Sn}\frac{4K}{9}=s_1, \qquad \operatorname{Sn}\frac{8K}{9}=s_2, \qquad \operatorname{Sn}\frac{12K}{9}=s_3 \quad \text{and} \quad \operatorname{Sn}\frac{16K}{9}=s_4.$$

Also let

$$x=s_1^2+s_2^2+s_3^2+s_4^2.$$

$$y=s_1^2s_2^2+s_1^2s_3^2+s_1^2s_4^2+s_2^2s_3^2+s_2^2s_4^2+s_3^2s_4^2.$$

$$z=s_1^2s_2^2s_3^2+s_1^2s_2^2s_4^2+s_1^2s_3^2s_4^2+s_2^2s_3^2s_4^2.$$

$$w=s_1^2s_2^2s_3^2s_4^2.$$

* By a method similar to that given in this paper Dr. Paul Mayer solved the problem of the division of the lemniscate into seven equal parts. See his doctorate dissertation "Ueber Siebenteilung der Lemniscate" (Strassburg, 1900) The division into nine equal parts is of interest as this is after Mayer's case the simplest case of a division which cannot be performed by ruler and compass alone.

Then s_1^2, s_2^2, s_3^2, s_4^2 are the roots of the equation

$$X^4 - xX^3 + yX^2 - zX + w = 0.$$

Now

$$\mathrm{Sn}\frac{12\,\mathrm{K}}{9} = \mathrm{Sn}\frac{2\mathrm{K}}{3},$$

and its value is known from the division of the lemniscate into three equal parts.

Let us therefore write

$$\mathrm{Sn}^2\frac{12\,\mathrm{K}}{9} = s_3^2 = a$$

and $\quad x_1 = s_1^2 + s_2^2 + s_4^2.$

$$y_1 = s_1^2 s_2^2 + s_2^2 s_4^2 + s_4^2 s_1^2.$$

$$z_1 = s_1^2 s_2^2 s_4^2$$

so that s_1^2, s_2^2, s_4^2 can be obtained from the solution of a cubic equation.

We know that

$$(1) \quad sn(u+v)\, sn(u-v) = \frac{sn^2 u - sn^2 v}{1 - k^2 sn^2 u sn^2 v}.$$

$$(2) \quad cn(u+v)\, cn(u-v) = \frac{cn^2 u\, cn^2 v - k'^2 sn^2 u\, sn^2 v}{1 - k^2 sn^2 u sn^2 v}$$

$$(3) \quad dn(u+v)\, dn(u-v) = \frac{dn^2 u\, dn^2 v + k^2 k'^2 sn^2 u\, sn^2 v}{1 - k^2 sn^2 u sn^2 v}.$$

Let us put $k^2=-1$ and substitute for u and v in (2) and (3) the following values in succession :

$$u=\frac{4K}{9};\ \frac{4K}{9};\ \frac{4K}{9};\ \frac{8K}{9};\ \frac{8K}{9};\ \frac{12K}{9}.$$

$$v=\frac{8K}{9};\ \frac{12K}{9};\ \frac{16K}{9};\ \frac{12K}{9};\ \frac{16K}{9};\ \frac{16K}{9}.$$

By making use of the equations (1), (2) and (3) we then get the following equations :

$$\phi_1=(1-a)(1-s_1^2)(1+s_1^2s_2^2)^2-\{1-s_1^2-s_2^2-s_1^2s_2^2\}^2=0.$$

$$\phi_2=(1-s_1^2)(1-s_2^2)(1+as_4^2)^2-\{1-a-s_4^2-as_4^2\}^2=0.$$

$$\phi_3=(1-s_1^2)(1-s_4^2)(1+as_2^2)^2-\{1-a-s_2^2-as_2^2\}^2=0$$

$$\phi_4=(1-a)(1-s_2^2)(1+s_2^2s_4^2)^2-\{1-s_2^2-s_4^2-s_2^2s_4^2\}^2=0.$$

$$\phi_5=(1-s_2^2)(1-s_4^2)(1+as_1^2)^2-\{1-a-s_1^2-as_1^2)^2=0.$$

$$\phi_6=(1-a)(1-s_4^2)(1+s_1^2s_4^2)^2-\{1-s_1^2-s_4^2-s_1^2s_4^2\}^2=0.$$

$$\psi_1=(1+a)(1+s_1^2)(1+s_1^2s_2^2)^2-\{1+s_2^2+s_2^2-s_1^2s_2^2\}^2=0.$$

$$\psi_2=(1+s_1^2)(1+s_2^2)(1+as_4^2)^2-\{1+a+s_4^2-as_4^2\}^2=0.$$

$$\psi_3=1+s_1^2)(1+s_4^2)(1+as_2^2)^2-\{1+a+s_2^2-as_2^2\}^2=0.$$

$$\psi_4=(1+a)(1+s_2^2)(1+s_2^2s_4^2)^2-\{1+s_2^2+s_4^2-s_2^2s_4^2\}^2=0.$$

$$\psi_5=(1+s_2^2)(1+s_4^2)(1+as_1^2)^2-\{1+a+s_1^2-as_1^2\}^2=0.$$

$$\psi_6=(1+a)(1+s_4^2)(1+s_1^2s_4^2)^2-\{1+s_1^2+s_4^2-s_1^2s_4^2\}^2=0.$$

$$\sum_{s=1}^{6}(\phi_s+\psi_s)$$

$$=-6(a^3+s_1^4+s_2^4+s_4^4)+6(s_1^2s_2^2+s_1^2s_4^2+s_2^2s_4^2+as_1^2$$

(4) $$+as_2^2+as_4^2)+12as_1^2s_2^2s_4^2+4a(s_1^4s_2^2+s_2^4s_4^2+s_4^4s_1^2)$$

$$+2\{a(s_1^6s_2^4+s_2^6s_4^4+s_4^6s_1^4)+a^2(s_1^2s_2^2s_4^4+s_1^2s_4^2s_2^4$$

$$+s_2^2s_4^2s_1^4)\}=0.$$

$$\phi_2+\phi_3+\phi_5+\psi_2+\psi_3+\psi_5$$

(5) $$=-6a^2-2x_1^2+6y_1+4ax_1+12az_1+2a^2z_1x_1=0$$

$$-\phi_2-\phi_3-\phi_5+\psi_2+\psi_3+\psi_5$$

(6) $$=-12a+4a^2x_1+4ax_1^2+2a^2x_1y_1-6a^2z_1=0.$$

$$a(\phi_1+\phi_4+\phi_6-\psi_1-\psi_4-\psi_6)+(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5+\phi$$

$$+\psi_1+\psi_2+\psi_3+\psi_4+\psi_5+\psi_6)$$

(7) $$=-12a^2-6x_1^2+18y_1+12ax_1-4a^2y_1-2a^2y_1^2+24az_1$$

$$-4ax_1y_1+6a^2x_1z_1$$

$$=0.$$

$$s_1^2s_2^2(\phi_2+\psi_2)+s_1^2s_4^2(\phi_3+\psi_3)+s_2^2s_4^2(\phi_5+\psi_5)$$

(8) $$=2y_1^2+12az_1-6x_1z_1+4ay_1z_1+6a^2z_1^2-2a^2y_1$$

$$=0.$$

$$s_1^2s_2^2(\psi_2-\phi_2)+s_1^2s_4^2(\psi_3-\phi_3)+s_2^2s_4^2(\psi_5-\phi_5)$$

(9) $$=-4ay_1-18z_1-2x_1y_1+12a^2z_1+12az_1x_1+4a^2y_1z_1$$

$$=0.$$

All these equations are satisfied by the following values of x_1 and y_1, as can be seen by actual substitution :

$$(10)\quad x_1=a^2z_1+2a.$$

$$(11)\quad y_1=a^2-2az_1.$$

The values of a are given by

$$(12)\quad a^4+6a^2-3=0.$$

Again, by forming the product of ψ_1, ψ_4, ψ_6, we have,

$$(1+a)^8(1+x_1+y_1+z_1)(1+y_1+x_1z_1+z_1^2)^2$$

$$(13)\quad =\{1+2x_1-4z_1+2y_1z_1-y_1^2+x_1^2-z_1^2\}^2$$

Substituting for x_1 and y_1, the equation becomes

$$(1+a)^8\{(1+a)^2+(1-a)^2z_1\}(1+a^2)^2(1+z_1^2)^2$$

$$(14)\quad =\{(1-a^4+4a+4a^2)+4(2a^3+a^2-1)z_1$$

$$-(1+4a+4a^2-a^4)z_1^2\}^2.$$

Expanding and neglecting a common factor z_1, the equation reduces to the quartic equation

$$(15)\quad z_1^4+6z_1^2+32\frac{a(1+a^2)}{(1-a^2)^2}z_1-3=0.$$

From (10), (11) and (15) we can get x_1, y_1 and z_1 and we can solve the cubic equation

$$(16)\quad X^3-x_1X^2+y_1X-z_1=0,$$

whose roots are s_1^2, s_2^2 and s_4^2.

The biquadratic equation has got one positive root lying between 0 and $\frac{4}{3}$, where

$$a = +(2\sqrt{3}-3)^{\frac{1}{2}}.$$

With this value of z_1, we easily see that x_1 and y_1 are positive quantities and the cubic equation (16) has got all its roots positive and real, and these are consequently the values of

$$Sn^2 \frac{4K}{9},\ Sn^2 \frac{8K}{9} \text{ and } Sn^2 \frac{16K}{9}.$$

In conclusion I wish to express my indebtedness to Prof. Ganesh Prasad, D.Sc., who kindly suggested the investigation to me and took great interest in the preparation of the paper.

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Early history of the arithmetic of zero and infinity in India

BY

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Introductory.

The arithmetic of zero is entirely the Hindu contribution to the development of the mathematical science. With no other early nations do we find any treatment of zero.* In a former paper, † I have shown that the zero was well known to the Brāhmaṇas of India in the second century B.C. The date of its actual invention must be put a few centuries earlier. Further an instance of the existence of zero has been traced even in the Atharvaveda. In certain Babylonian records of about 200 B.C., we find the use of a character to denote the absence of a member.‡ But that did not form the basis of their system of numeration as does the Hindu zero. The Mayas of Central America had a system of recording dates (on the scale of 20, except in one step), in which they used a symbol for the zero. But the system was not used for

* This fact deserves the most careful consideration of those who claim a non-Hindu origin for our modern system of numerals.

† Bibhutibhusan Datta, "Early literary evidence of the use of the zero in India," *American Math. Monthly*, Vol. 33, 1926, pp. 449-454.

‡ It should be noted that though we possess records of the Babylonian sexagesimal notation dating from as early as 2400-1600 B.C., it is only in the records of about 200 B.C, that we find a symbol for the absence of a figure. The Hindus also had a symbol for zero, near about the same time, if not earlier So it will be futile to suggest, as has been done by Professor Florian Cajori (*History of Mathematics*; New York, 1922, p. 88), that there is a Babylonian influence in the Hindu system of decimal numeration. Therefore his name "Babylonic-Hindu" for the modern place value system of decimal notation will be as much a misnomer as the present "Arabic" or "Hindu-Arabic" is. It should be properly called simply "Hindu".

purposes of ordinary computation. Nor does the earliest of these records is known to date from much before the beginning of the Christian era.* Thus for our modern conception of zero and for its use in a fully developed and highly ingeneous system of decimal notation entirely hinging upon it, the world is indebted to the Hindus. Hence it will be natural to expect to find the earliest instances of treatment of zero amongst them. All the known mathematical treatises of the Hindus, with the single exception of the treatise of Āryabhaṭa (born 476 A.D.), have a section devoted to the subject under consideration. The Hindus speak of all the fundamental arithmetical operations in relation to zero. And except in the case of division of a finite quantity by zero, all the results were correctly obtained from the beginning. The quotient of division by zero, has been correctly stated for the first time by Bhāskarācārya (born 1114 A.D.).

It should be observed that almost all the Hindu mathematicians have mentioned separately of operations with zero and by zero.

Definition of Zero.

It has been stated by Brahmagupta (628 A.D.) that the sum of two equal and opposite quantities is zero (*śūnya*).†

$$(+a)+(-a)=0, \quad \text{or} \quad a-a=0.$$

The same statement has been repeated by all the successive Hindu mathematicians. In this sense zero signifies the absence of a quantity—*nothing*. A similar definition of zero was given in the nineteenth century by Martin Ohm of Berlin (1828) and Wolfang Bolyai de Bolya of Hungary (1832).‡ Most of the commentators of Bhāskara define zero as *abhāva*, which literally means 'want or absence', meaning of course the absence of a quantity.

When defined in this way, zero cannot either operate upon a quantity or be operated upon. For all operations imply the existence of the quantities concerned. This idea is fully borne out by a certain

* Cajori, *loc. cit.*, p. 69.

† *Brāhma-sphuṭa-siddhānta*, ed. Sudhakara Dvivedi, Benares, 1902, ch. xviii, verse 31.

‡ For these and other references to the history of operations with zero in Europe, I am indebted to the paper of Mr. H. G. Romig, "Early History of Division by Zero," in the *American Math. Monthly*, Vol. 31, 1924, pp. 387-9.

remark of Kriṣṇa (1575). In giving a proof of multiplication of zero, he says: "In fact multiplication is repetition: and if there be nothing to be repeated, what should the multiplicator repeat, however great it be?"* Hence truly speaking the expressions $0 \times a$, $0/a$ and $a/0$ are meaningless.

It is, however, possible to assign certain meanings to those symbols. For this purpose zero is given a significance different from absolute nothing. It is conceived as the limiting value of a continuously decreasing sequence. Thus according to Kriṣṇa as well as Gaṇeśa (1545) "the utmost diminution of a quantity is the same with the reduction of it to nothing." This definition of zero is further confirmed by Raṅganātha (1602) who considers the zero to be a quantity in the utmost degree small. In this way zero is, indeed, defined as an *infinitesimal*. And this is our modern conception of zero.

If regarded as absolute nothing zero cannot be called a number. Thus John Wallis (1657) and Bishop Berkeley (1707) declare zero to be no number. On the other hand the Hindu mathematicians consider it to be a number. This has been particularly specified by Nrisiṃha (1621) who declares that "though the cipher (*śūnya*) is the blank (*abhāva*), it denotes a *number* and is therefore included amongst the numerals."† In 850 A.D., Mahāvīrācārya calls it a *saṃkhyā (number)* in the same sense as he calls the nine significant figures 1 to 9.‡

Up to this time we have confined ourselves to the mathematical treatises only. Some of the definitions of zero found in earlier non-mathematical works are very succint and accurate, even according to our modern conception. In the Atharvaveda, the zero is called *kṣudra* (trifling). In the *Amarakoṣa*, a noted Sanskrit lexicon of the 5th century A. D., it is called *tuccha* (insignificant, negligible), in addition to *abhāva* (blank), *asampūrṇa* (incomplete) and *ūna* (less).

It has been remarked by Kriṣṇa that "Cipher is neither positive nor negative: it is therefore exhibited with no distinction of sign. No difference arises from the reversing of it and none is here shown."

* Henry T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*, London, 1817, p. 137, footnote 2.

† Vide *Siddhānta-Śiromaṇi*, edited by Muralidhara Jha with Munīśvara's *Marīci* and Nrisiṃha's *Vāsanā-vārtika*, Benares, 1917, p. 89.

‡ Mahāvīrācārya, *Gaṇita-sāra-Saṃgraha*, edited and translated into English by M. Rangacarya, Madras, 1912, Sanskrit Text, p. 7.

Addition und Subtraction.

The addition and subtraction of the zero are first found incidentally mentioned in Varāhamihira's astronomical treatise, *Pañca-siddhāntikā* (505 A.D.).*

$$a \pm 0 = a.$$

And they have been restated in all the succeeding mathematical treatises of India.

Brahmagupta (628 A.D.) and Bhāskara (b. 1114 A.D.) have noted the changes of sign, if any, in the course of the operations. Thus

$$(\pm a) \pm 0 = \pm a; \qquad 0 + (\pm a) = \pm a;$$

$$0 \pm 0 = 0; \qquad 0 - (\pm a) = \mp a.$$

None of these results, probably except the last, requires any demonstration. About the last result, Kriṣṇa (1575 A.D.) remarks that subtraction diminishes a quantity by so much as is the amount of the subtrahend; if the quantity be reduced, the result of the subtraction is diminished accordingly: if reduced to nought, the result is diminished to its greatest degree: the amount of the subtrahend with the sign changed.†

Multiplication, Involution and Evolution.

The multiplication of cipher is found correctly stated in the works of all Indian mathematicians from Brahmagupta (628 A.D.) onwards:

$$0 \times (\pm a) = 0; \qquad (\pm a) \times 0 = 0; \qquad 0 \times 0 = 0.$$

The proof has been supplied in the following words by the eminent mathematician Gaṇeśa (1545 A D.) in his commentary on Bhāskara's *Līlāvatī*. "Each time the multiplier is diminished by unity, the product is diminished by an amount equal to the multiplicand. So in the extreme for a number multiplied by the zero, the product will be diminished by itself, *i.e.*, it is zero." Obviously here Gaṇeśa assumes the multiplicator to have been an integer to begin with; so that on repeatedly subtracting unity from it, it will be reduced ultimately to

* *Pañca-siddhāntikā*, edited with English translation and Sanskrit commentary by G. Thibaut and Sudhakara Dvivedi, Benares, 1899, ch. VIII, verses 2, 17; iv. 7, 8, 11, 12; xviii 35, 44, 45. *cf.* Bibhutibhusan Datta, *loc. cit.*, p 452.

† Colebrooke, *loc. cit*, p. 136, fn. 4.

the zero. The proof of Kriṣṇa, though substantially similar, is free from all such limitations. "The more the multiplicand is diminished, the smaller is the product; and if it be reduced in the utmost degree, the product is so likewise: now the utmost diminution of a quantity is the same with the reduction of it to nothing: therefore, if the multiplicand be nought, the product is cipher. In like manner, as the multiplier decreases, so does the product; and, if the multiplier be nought, the product is so too." *

Brahmagupta states the value of the square and square root of the zero; Bhāskara adds the cube and cube root as well.

$$0^2=0; \quad 0^3=0; \quad \sqrt{0}=0; \quad \sqrt[3]{0}=0.$$

Division.

The quotient of division of the zero by a finite quantity has been correctly given by all the writers:

$$\frac{0}{a}=0.$$

Kriṣṇa says: "If the dividend be diminished, the quotient is reduced and if the dividend be reduced to nought, the quotient becomes cipher."

In the case of division by zero, most of the early writers have stumbled and failed to give the quotient correctly for several centuries. It has been so in the Occident as in the Orient.

The first writer to speak of division by zero is Brahmagupta (628 A.D.). Srīdhara (c. 750 A.D.) and Āryabhaṭa II (c. 950 A.D.) do not mention of division by zero. Mahāvīra (c. 850 A.D.) has only an incorrect result: "that (number) remains unchanged when it is divided by zero."† Probably he interprets that a division by zero, which is nothing, is no division at all.

It has been stated by Brahmagupta that "cipher, divided by cipher, is nought" but a "positive or negative (quantity), divided by cipher is *taccheda*."‡ Now, *taccheda* is a technical term which literally means 'having that for denominator'; having in this instance cipher

* Colebrooke, *loc. cit.*, p. 137, fn. 2.

† *Gaṇita-sāra-Saṃgraha*, Ch. i, verse 49.

‡ *Brāhma-sphuṭa-siddhānta*, Ch. xviii, verse 35.

(*kha*) as denominator, it is equivalent to, and has been obviously meant for, *kha-cheda*. Thus, according to Brahmagupta

$$\frac{0}{0}=0; \quad \frac{\pm a}{0}=\text{kha-cheda}.$$

The first result is wrong; for the value of 0/0 is indeterminate, *not* zero. Bhāskara nowhere expressly mentions of the division of zero by zero; nor do Śrīdhara, Mahāvīra and Āryabhaṭa II. It, however, appears from the writings of Bhāskara that he considers the quotient to be different from what has been stated by Brahmagupta.

Besides giving a special technical name, Brahmagupta does not state what will be the true arithmetical value of the quotient $a \div 0$. The necessity for a new special nomenclature, however, abundantly testifies that he must have been aware of its possessing certain attribute which is not possessed by an ordinarily known number. The true value is found first in the works of Bhāskara who calls the quotient by an alike technical name *kha-hara*. "A finite quantity, divided by cipher, is *kha-hara*," says Bhāskara.* The value of *kha-hara* is given as infinite (*anantarāśi*).†

"As much as the divisor is diminished," says Krisṇa, "so much is the quotient raised. If the divisor be reduced to the utmost, the quotient is to the utmost increased. But, if it can be specified that the amount of the quotient is so much, it has not been raised to the utmost: for a quantity greater than that can be assigned The quotient is indefinitely great, and is rightly termed infinite." Raṅganātha gives a similar proof.

Ata Ulla Rashidi, an Indian Mohammedan who translated Bhāskara's *Vijagaṇita* into Persian in 1634, thinks it impossible to divide a finite quantity by zero.‡ "If the dividend is cipher and the divisor

* Compare *Līlāvatī*: "Kha-bhājito rāśiḥ kha-hara syāt;" *Vījagaṇita*: "Kha-haro bhavet khena bhaktaśca rāśi."

Cheda=*hara*=*hāra*=denominator; hence the terms of Brahmagupta and Bhāskara are practically the same. The latter has been admittedly influenced by the former, in writing his mathematical and astronomical treatises.

† *Vījagaṇita*, ed. Sudhakara Dvivedi, Benares, 1888, pp. 6-7.

‡ An account of this translation has been published by Edward Strachey as *Bija-Gaṇita* (London. 1818); this translation "does not in itself afford a correct idea of its original, as a translation should do; for it is an undistinguished mixture of text and commentary, and in some places it even refers to Euclid." It can, however, be taken as fairly representing the knowledge of the Persians on the subject under discussion, for Ata Ulla Rashidi was certainly well acquainted with the Persian mathematical treatises. Strachey's account is "partly literal translation, partly abstract" and partly in his own words.

a number the quotient will be cipher. For example if we divide cipher by 3, the quotient will be cipher, for multiplying it by the divisor the product will be the dividend, which is cipher and if a number is the dividend and cipher the divisor, the division is impossible; for by whatever number we multiply the divisor, it will not arrive at the dividend, because it will always be cipher "*

With the above may be compared the remark of Martin Ohm (1828 A.D.)†, who says that " if a is not zero, but b is zero, then the quotient a/b has no meaning ;" for the quotient " multiplied by zero gives only zero and not a, as long as a is not zero."‡ Apparently Martin Ohm thinks, like Brahmagupta, that $0/0=0$, which is wrong. According to Wolfang Bolyai de Bolya (1832 A.D.), " 1/0 is an impossible quantity," but he considers that " if z tends towards 0, then $1/z$ tends towards infinity."‡.

Evaluation of $a\times 0\div 0$.

There is a certain passage in Bhāskara's *Līlāvatī*, of which the true significance is not quite clear. Bhāskara says, "When a number is multiplied by cipher, the product is cipher; but in case any operation remains to be done, cipher is merely conceived to be multiplier; for when cipher is the multiplier, and cipher also becomes the divisor the number is considered unchanged."§ Thus according to him

$$a\times 0=0\,;$$

but $$(a\times 0)\div 0=a.$$

There is also an instance of the kind

$$(a\div 0)\times 0=a.$$

Apparently these last two are expressions are of the type $0/0$ and $\infty\times 0$, which are indeterminate, *not* equal to a finite quantity as has been assumed by Bhāskara. Thus they appear to be cases of confusion and blunder on the part of their author. But it should be pointed out that

* Strachey, *loc. cit.*, pp. 4-5. This passage is a correct literal rendering of Rashidi's Persian original.

† Martin Ohm, *Lehrbuch der niedern Analysis*, Vol. I, Berlin, 1828, pp. 110, 112; *Der Geist der Differential-und Integral-Rechnung*, Erlangen, 1846, pp. 18, 76.

‡ Wolfang Bolyai de Bolya, *Tentamen*, 2nd ed., Vol. I, Budapest, 1897. § 32, p. 45; § 21, pp. 86, 91.

§ Līlāvatī, translated by John Taylor, Bombay, 1816, p. 29.

in proceeding to such a conclusion, we perform certain intermediate operations, *viz.*, $a\times 0=0$ and $a/0=\infty$, which Bhāskara admits to be correct and justified when considered apart, but which he has particularly warned us not to make in cases like the above. Bhāskara's commentators do not seem to have better understood the proper significance of the passage in question. The most eminent of them, Gaṇeśa, failed to bring more light to bear upon it. "Similarly, if cipher become the multiplier of a number, should there be further operations (to be done), then it should not be performed that a number multiplied by cipher is zero; but cipher should be set down by the side of that (*i.e.*, the multiplicant) as a multiplier. Then if in performing the subsequent operations, cipher happens to become a divisor, then the multiplier cipher and the divisor cipher, being equal, should be cancelled. Otherwise a number multiplied and divided by cipher will be equal to zero." * Obviously he had in mind an operation of the kind

$$(a\times\epsilon)\div\epsilon=a,$$

where ϵ is a small quantity of higher order which tends towards the limiting value 0. Taylor is of the same opinion.†

Like the term *kha-hara* for the quotient $a\div 0$, Bhāskara has a special technical term for the product $a\times 0$, namely, *kha-guṇa*, which literally means "that which has cipher as multiplier." Ordinarily the value of the product is zero, but $(kha\text{-}guṇa)/0$ is a finite quantity. Similarly $(kha\text{-}hara)\times 0$ is a finite quantity.

In the passage referred to, Bhāskara seems to have in mind to assign a meaning to the expressions of the forms $0/0$ and $\infty/0$. We have seen that the first expression is considered to be equivalent to zero by Brahmagupta. Bhāskara seems to have in mind to correct that statement of Brahmagupta and say that $0/0$ as well as $\infty\times 0$ may stand for any finite quantity. If this surmise be true, which appears to be very likely so, here is the first attempt to assign meanings to expressions of those types. After all, however, it will have to be admitted that the choice of language has not been happy and unambiguous.

In the opinion of Bapu Deva Sastri, the rule indicates a method of finding the true limiting values of certain functions (defined by a chain of successive operations) which involve the independent variable

* Gaṇeśa seems to be of the same opinion as Brahmagupta that $0/0=0$.

† *loc. cit.*, p. 30, footnote B.

in such a way as to lead to the one or the other of those indeterminate forms for a particular value of the variable.*

Though the true significance of the passage in question is not free from ambiguity, there is, however, no doubt about the fact that Bhāskara attached much importance to the principle underlying the same. For it has been remarked by him that that method of calculation is of much assistance in mathematical astronomy.† Bhāskara's mathematical treatises contain three illustrative examples. In our modern symbolism, they will be put in the following forms.

(*i*) $$\frac{\left(x\times 0+\frac{x\times 0}{2}\right)\times 3}{0}=63,$$

or $$\left(x+\frac{x}{2}\right)\times 3=63,$$

or $$x=14.$$

(*ii*) $$\left\{\left(\frac{x}{0}+x-9\right)^2+\left(\frac{x}{0}+x-9\right)\right\}\times 0=90,$$

or $$\left(\frac{x^2}{0}+\frac{x}{0}\right)\times 0=90,$$

or $$x^2+x=0,$$

or $$x=9.$$

(*iii*) $$\left[\left\{\left(x+\frac{x}{2}\right)\times 0\right\}^2+2\times\left\{\left(x+\frac{x}{2}\right)\times 0\right\}\right]\div 0=15,$$

or $$\left(\frac{9x^2}{4}\times 0^2+\frac{6x}{2}\times 0\right)\div 0=15,$$

or $$9x^2+6x=60,$$

$$\therefore\quad x=2.$$

* Bapu Deva Sastri, *Vijagaṇita (in Hindi)*, Part I. Benares, 1875. p. 179 *et. sq.*

† *Līlāvatī*, "asya gaṇitasya grahagaṇite mahānupayoga,"

Misapprehension cleared.

Thibaut is of opinion that the true arithmetical value of the quotient $a/0$ was not known to Bhāskara. The passage which is commonly found in all the available manuscripts of Bhāskara's *Vijagaṇita* and wherein the quotient has been expressly stated to be infinite (*ananta-rāśi*), is supposed by him to be an interpolation of certain commentator.* This misapprehension of Thibaut is also shared by D. E. Smith.† None of them have adduced any proof in support of their supposition. The passage in question occurs in the oldest known commentary of Bhāskara's *Vijagaṇita* and was known to his other commentators. Again immediately following that passage, it has been clearly stated by the author that the quotient $a/0$ is comparable only with the Infinite God. This comparison would not have any force, if the true value of $a/0$ had not been infinite in the estimation of Bhāskara. A little further on Bhāskara has applied the principle underlying this latter statement in working out a specific example [Ex., *ii* given above].‡ All these facts prove clearly and sufficiently that Bhāskara knew the value of the quotient $a/0$ to be infinite. Western historians of mathematics, such as Cantor, Hankel, and Fink and oriental scholars like Taylor, Colebrooke, Sudhakara Dvivedi, and Bapu Deva Sastri are of the same opinion.

Infinity.

We have already seen that, in Sanskrit, one of the terms for infinity is *kha-hara*, because it is, as the result of the division of a finite quantity by zero, that we get the idea of a quantity of infinite magnitude in mathematics. Thus

$$\text{Infinity} = \frac{a}{0}, \qquad a \neq 0.$$

This is, indeed, no formal definition of infinity. It is only a convention to give an idea of infinity by referring to one of its properties. We have seen that the expression *kha-cheda*; which is the same as *kha-hara*, existed from the time of Brahmagupta. But it was Bhāskara who first rightly attributed a property to that term. Now

* G. Thibaut, *Astronomie, Astrologie und Mathematic*, Strasbourg, 1899, p. 72.

† David Eugene Smith, *History of Mathematics*, Vol. I, Boston, 1923, pp. 277-8.

‡ Colebrooke, *loc. cit.*, p. 213.

this is exactly the modern convention regarding infinity. Burkhardt says: "In addition to the complex numbers and their symbols already introduced, we introduce now a new one, 'infinity,' with the symbol ∞, which is to be regarded as the result of the division 1/0." * It will be interesting to know in this connection that about the propriety of this convention of regarding infinity there has been carried on in recent years a heated controversy by some distinguished professors of mathematics in the pages of the *American Mathematical Monthly*.†

The classical methed of defining infinity is to regard it as the limiting value of a gradually increasing sequence. In a work of about 300 B.C., infinity (*ananta*) is described as a number as great as the number of grains of sand on the brink of all the rivers on the earth or the drops of water in the oceans.‡ In another work we get things far more interesting and valuable. It is said that a number which will be regarded as infinite (*ananta*) in relation to something, may be small (*anta*, literally finite) in relation to another.§ Or again a number may be divided into infinite parts, each of which can be again subdivided infinitely. Thus we get that *finite* and *infinite* are only relative terms.

Bhāskara says that infinity remains unaffected by the addition or subtraction of a finite quantity however large.

$$\frac{a}{0} \pm \beta = \frac{a}{0}$$

where a and β are finite. The proof adduced by him depends more upon the theological beliefs of the Hindus than upon any arithmetical reasoning. "In this quantity," says Bhāskara, "consisting of that which has cipher for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and

* Burkhardt, *Theory of Functions of a Complex Variable*, translated by Rasor.

† Compare *American Math. Monthly*, 1922, p. 293; 1923, pp. 255, 384; 1924, p. 383.

‡ *The Kalpa Sūtra and Nava Tattva*, English translation by J. Stevenson, London, 1848, p. 16.

§ *Uttarādhyāyanasūtra*; it is published in original by J. Charpentier, Upsala, 1922, and in English translation by H. Jacobi (*Gaina Sūtras*, S. B. E., Oxford, 1895), *vide* original xxiii. 17, 24; trans. pp. 195-6. This work is one of the *Mūlasūtras* of the Jainas and is believed to have been composed about 300 B.C.

immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth." *

The first true mathematical proof of the above comes from Gaṇeśa. Kriṣṇa observes: "This fraction (*kha-hara*) indicating an infinite quantity, is unaltered by addition or subtraction of a finite quantity. For, in reducing the quantities to a common denominator, both the numerator and denominator of the finite quantity, being multiplied by cipher, become nought: and a quantity is unaltered by addition or subtraction of nought. The numerator of the infinite fraction may indeed be varied by the addition or subtraction of a finite quantity, and so it may be that of another infinite fraction: but whether the finite numerator of a fraction, whose denominator is cipher be more or less, the quotient of its division by cipher is alike infinite." †

$$\frac{a}{0} \pm \frac{b}{c} = \frac{a}{0} \pm \frac{b \times 0}{c \times 0} = \frac{a \pm 0}{0} = \frac{a}{0}$$

b/c being finite. Again

$$\frac{a}{0} \pm \frac{b}{0} = \frac{a \pm b}{0} = \text{infinite}.$$

This last result is incorrect for it is truly indeterminate.

* Colebrooke, *loc. cit.*, p. 138.

† *Ibid*, p. 137, footnote 5.

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3

On the summability (C1) of the Legendre series of a function at a point where the function has a discontinuity of the second kind.

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The chief object of the present paper is to formulate a definite answer to the question: * Does Fejér's mean *always* converge, with increasing n, in the case of the Legendre series of a function when the Fejér's mean of the Fourier series of the same function converges? I show by examples that the answer is in the negative.

It is believed that this difference in the convergence properties of the two series was not noticed by any previous writer.

I consider also the behaviour of Fejér's mean at a point of discontinuity of the second kind of the function at some length as I have done in my paper relating to Fourier series.†

So much has been written on the subject of the Legendre series that I can safely take for granted certain results as well-known.‡ Thus, the

* In the abstract of Prof. W. H. Young's paper, "On the connexion between Legendre series and Fourier series," it is stated that "if Cesàro convergence be considered instead of ordinary convergence, corresponding results hold *mutatis mutandi*. Thus, for example, in the case of Legendre series, the Cesàro convergence properties are the same as those of Fourier series" (*Proc. L. M. S.*, Vol. 17, p. XI). It is very likely that, in making this statement, Prof. Young had in his mind only continuous functions and those having ordinary discontinuities. The paper is published in Vol. 18 of the *Proceedings* but does not treat the case of Cesàro convergence explicitly.

† See No. 3 of this volume of the *Bulletin*.

‡ See *e.g.*, Hobson's papers in Vols. 6 and 7 of the *Proc. L. M. S., Ser. 2*, Gronwall's papers in Vols. 74 and 75 of the *Math. Ann.* and Haar's papers in *Rendiconti del Circolo Mat. di Palermo*, Vol. 32 and *Math. Ann.*, Vol. 78.

Fejér's mean $S_n(x)$ of any series $\sum u_n(x)$ being

$$\frac{1}{n+1} \sum_{\nu=0}^{n} (n-\nu+1)u_\nu,$$

for the Legendre series of $f(x)$, *i.e.*, the series

$$\sum_{n=0}^{\infty} \frac{(2n+1)}{2} P_n(x) \int_{-1}^{1} P_n(\mu) f(\mu)\, d\mu,$$

$S_n(x)$ is

$$\frac{1}{2(n+1)} \int_{-1}^{1} f(\mu)\Omega_n(x, \mu) d\mu,$$

where $\Omega_n(x, \mu)$ denotes

$$\sum_{\nu=0}^{n} (n-\nu+1)(2\nu+1)P_\nu(x)P_\nu(\mu).$$

Another result which is known is, that for an absolutely integrable function the behaviour of $S_n(x)$ at a point x_0 inside the interval $(-1, 1)$ depends only on the values of $f(x)$ in a neighbourhood of x_0 as small as we please, so that for the purpose of this paper we need only consider the behaviour of

$$\frac{1}{2(n+1)} \int_{x_0-\epsilon}^{x_0+\epsilon} f(\mu)\Omega_n(x_0, \mu) d\mu,$$

where ϵ is an arbitrarily small quantity greater than 0.

Throughout this paper, $\mu = \cos\theta$, $x = \cos\phi$, $x_0 = \cos\phi_0$ and $\chi(z)$ and $\psi(z)$ are monotone functions which tend to infinity as z tends to 0.

1. I first proceed to prove that $S_n(\cos\phi_0)$ *behaves as*

$$\frac{1}{\pi} \int_0^{\epsilon_1} \frac{\sin (n+1)z}{z} [f\{\cos(z+\phi_0)\} + f\{\cos(-z+\phi_0)\}] dz.$$

Proof. Since

$$\sum_{\nu=0}^{n} (2\nu+1)P_\nu(x)P_\nu(\mu) = (n+1) \frac{P_{n+1}(x)P_n(\mu) - P_n(x)P_{n+1}(\mu)}{x-\mu},$$

$$\Omega_n(x,\mu)\equiv(n+1)\sum_{\nu=0}^{n}(2\nu+1)P_\nu(x)P_\nu(\mu)-\sum_{\nu=1}^{n}\nu(2\nu+1)P_\nu(x)P_\nu(\mu)$$

$$=(n+1)^2.\frac{P_{n+1}(x)P_n(\mu)-P_n(x)P_{n+1}(\mu)}{x-\mu}$$

$$-\sum_{\nu=1}^{n}\nu(2\nu+1)P_\nu(x)P_\nu(\mu).$$

Now it can be shown that $(2\nu+1)P_\nu(x)P_\nu(\mu)$ is numerically less than a finite quantity D for every value of ν; hence

$$\left|\sum_{\nu=1}^{n}(2\nu+1)\nu P_\nu(x)P_\nu(\mu)\right|< D\sum_{\nu=1}^{n}\nu, \text{ i.e., } Dn,$$

Therefore the part of

$$\frac{1}{2(n+1)}\int_{x_0-\epsilon}^{x_0+\epsilon}f(\mu)\Omega_n(x_0,\mu)d\mu$$

due to $\sum_{\nu=1}^{n}\nu(2\nu+1)P_\nu(x_0)P_\nu(\mu)$ is numerically less than

$$\tfrac{1}{2}D\int_{x_0-\epsilon}^{x_0+\epsilon}|f(\mu)|\,d\mu,$$

which, because of the absolute integrability of $f(\mu)$, tends to zero with ϵ.

Thus this part may be neglected.

Again, it is well-known that

$$P_n(\mu)=\frac{2}{\sqrt{\pi}}\frac{\Pi(n)}{\Pi(n+\frac{1}{2})}\Bigg[\frac{\cos\left\{(n+\frac{1}{2})\theta-\frac{\pi}{4}\right\}}{(2\sin\theta)^{\frac{1}{2}}}+\frac{1^2}{2.(2n+3)}$$

$$\times\frac{\cos\left\{(n+\frac{3}{2})\theta-\frac{3\pi}{4}\right\}}{(2\sin\theta)^{\frac{3}{2}}}+\frac{1^2\cdot3^2}{2.4.(2n+3)(2n+5)}\left\{\frac{\cos(n+\frac{5}{2})\theta-\frac{5\pi}{4}}{(2\sin\theta)^{\frac{5}{2}}}\right\}$$

$$+\ \dots\ \dots\Bigg]$$

Therefore

$$(n+1)^2\frac{P_{n+1}(x)P_n(\mu)-P_n(x)P_{n+1}(\mu)}{x-\mu}$$

$$=\frac{(n+1)}{\cos\theta-\cos\phi}\cdot\frac{2}{\pi}\cdot\frac{1}{\sqrt{\sin\theta\sin\phi}}$$

$$\times\left\{\cos\left(\frac{2n+3}{2}\theta-\frac{\pi}{4}\right)\cos\left(\frac{2n+1}{2}\phi-\frac{\pi}{4}\right)\right.$$

$$\left.-\cos\left(\frac{2n+1}{2}\theta-\frac{\pi}{4}\right)\cos\left(\frac{2n+3}{2}\phi-\frac{\pi}{4}\right)\right\}$$

$$+\frac{O_n}{(\sin\theta\sin\phi)^{\frac{3}{2}}}, \qquad \ldots \quad (1)$$

where O_n is bounded for every n, $0<\theta<\pi$, $0<\phi<\pi$.

It is easily seen that the expression within the crooked brackets in the right side of (1) is

$$\sin(n+1)(\phi-\theta)\sin\frac{\theta+\phi}{2}-\cos(n+1)(\theta+\phi)\sin\frac{\phi-\theta}{2}.$$

Therefore the right side of (1) is equal to

$$\frac{(n+1)}{\pi\sqrt{\sin\theta\sin\phi}}\left\{\frac{\sin(n+1)(\theta-\phi)}{\sin\frac{\theta-\phi}{2}}-\frac{\cos(n+1)(\theta+\phi)}{\sin\frac{\theta+\phi}{2}}\right\}$$

$$+\frac{O_n}{(\sin\theta\sin\phi)^{\frac{3}{2}}}.$$

Hence

$$\frac{1}{2(n+1)}\int_{x_0-\epsilon}^{x_0+\epsilon}f(\mu)\Omega_n(x_0,\mu)d\mu$$

$$=\int_{x_0-\epsilon}^{x_0+\epsilon}\frac{f(\mu)}{2\pi\sqrt{\sin\theta\sin\phi_0}}\cdot\frac{\sin(n+1)(\theta-\phi_0)}{\sin\frac{\theta-\phi_0}{2}}d\mu$$

+ terms which may be neglected.

But $\displaystyle\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{f(\mu)}{2\pi\sqrt{\sin\theta\sin\phi_0}} \quad \frac{\sin(n+1)(\theta-\phi_0)}{\sin\frac{\theta-\phi_0}{2}}\,d\mu$

$$=\frac{1}{\pi}\int_0^{\epsilon_1}\frac{\sin(n+1)z}{z}\Big[f\Big\{\cos(z+\phi_0)\Big\}+f\Big\{\cos(-z+\phi_0)\Big\}dz\Big],$$

putting $\theta-\phi_0=z$, where ϵ_1 is an arbitrarily small quantity which is greater than 0 and depends on ϵ.

Case: $f\{\cos(z+\phi_0)\}+f\{\cos(-z+\phi_0)\}=\cos\psi(z)$

2. In this case $S_n(x_0)$ behaves as

$$\frac{1}{\pi}\int_0^{\epsilon_1}\frac{\sin(n+1)z}{z}\cos\psi(z)\,dz.$$

But, in his famous memoir,* Du Bois-Reymond has proved that with increasing n the above integral tends to a limit or not, according as

$$\psi \succ \log\frac{1}{z^2}\text{, or not.}$$

Thus, it is proved that the Fejér's mean of a Legendre series converges or diverges together with that of the Fourier series † corresponding to the same function, if that function has a *finite* discontinuity of the type of $\cos\psi(z)$.

The fact that in the case of Fourier series the mean behaves as

$$\frac{2}{n\pi}\int_0^{a}\Big(\frac{\sin nz}{z}\Big)^2\cos\psi(z)\,dz,$$

where a is an arbitrarily small quantity, does make only this difference, noticed by Fejér‡, that in the case of divergence, the mean for the Fourier series lies between the upper and lower limits of the function whilst the mean for the Legendre series does *not* lie between those limits.

* "Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungsformeln," p. 37. (*Abhandlungen der k. bayer. Akademie der Wissenschaften*, Vol. XII, 1876.)

† See my paper in No. 8 of this volume.

‡ See *Math. Ann.*, Bd. 67.

Case: $f\{\cos(z+\phi_0)\}+f\{\cos(-z+\phi_0)\}=\chi(z)\cos\psi(z)$

3. In this case $S_n(x_0)$ behaves as

$$\frac{1}{\pi}\int_0^{z_1}\frac{\sin(n+1)z}{z}\chi(z)\cos\psi(z)dz.$$

But for this case Du Bois-Reymond's result* is that the limit of the integral exists or not, according as

$$\psi(z)\succ\log\frac{1}{z^2}\text{ and }\chi(z)\prec z\sqrt{\psi''(z)},$$

or
$$\psi(z)\precsim\log\frac{1}{z^2}.$$

If $\psi(z)\succ\log\frac{1}{z^2}$ and $\chi(z)\succ z\sqrt{\psi''(z)}$, the limit is non-existent.

Thus, for $\chi(z)=z^{-\frac{1}{2}}$, $\psi(z)=\frac{1}{z}$,

$S_n(x_0)$ tends to a limit.

Some Cases in which $\lim_{n=\infty} S_n(\cos\phi_0)$ *is non-existent.*

4. (*a*) Let $\chi(z)=\left(\log\frac{1}{z^2}\right)^{\frac{1}{3}}$, $\psi(z)=\left(\log\frac{1}{z^2}\right)^{\frac{3}{2}}$.

Then $$\chi(z)\succ z\sqrt{\psi''(z)}.$$

Therefore $\lim_{n=\infty} S_n$ is non-existent.

(*b*) Let $\chi(z)=\left(\log\frac{1}{z^2}\right)^m$, $\psi(z)=\left(\log\frac{1}{z^2}\right)^{1+k}$, where $m>0$, $k>0$,

Then, if $m>\frac{k}{2}$, $\chi(z)\succ z\sqrt{\psi''(z)}$,

and the limit is non-existent.

(*c*) Let $\chi(z)=\frac{1}{z^m}$, $\psi(z)=\frac{1}{z^k}$, where $m>0$, $k>0$.

* *Loc. cit.*

$$\text{Then, if } m > \frac{k}{2},\ \chi(z) \succ z\sqrt{\psi''(z)}$$

and the limit is non-existent.

Behaviour of Fejér's mean for Fourier series.

5. I proceed to prove that for the case (a), considered above, the Fejér's mean of the Fourier series tends to a limit.

Proof:—

According to Prof. W. H. Young,* the derived series of the Fourier series of a function $F(x)$ converges (Cl) at a point x_0, if one of the derivates of $F(x)$ is known to be (1) summable in a neighbourhood of x_0 and, except possibly at a countable set of points, finite and (2)

$$\operatorname*{Lim}_{h=0} \frac{F(x_0+h)-F(x_0-h)}{2h}$$

exists; the sum of the derived series is this limit.

Now take

$$F(x) = \int_0^x f(t)dt;$$

then the Fourier series of $F(x)$ is easily seen to be

$$\frac{1}{2} A_0 + \sum_{n=1}^{\infty} \frac{1}{n} (a_n \sin nx - b_n \cos nx) - a_0 \sum_1^{\infty} \frac{(-1)^n}{n} \sin nx,$$

where $A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt,\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\ dt,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt$

Thus the Fourier series of the function

$$G(x) \equiv F(x) - \frac{a_0 x}{2} - \frac{1}{2} A_0$$

is
$$\sum_{n=1}^{\infty} \frac{1}{n} (a_n \sin nx - b_n \cos nx),$$

* "On the convergence of the derived series of Fourier series" (*Proc. L. M. S.*, Vol. 17, p. 230).

and therefore the derived series of the Fourier series of $G(x)$ is

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

But $F'(x)$ exists * even at the point of discontinuity x_0 of $f(x)$ and equals zero. Therefore

$$\operatorname*{Lim}_{h=0} \frac{G_0(x+h)-G_0(x-h)}{2h} = -\frac{1}{2} a_0.$$

Also, in the neighbourhood of x_0, $G'(x)$, being $f(x)-\frac{1}{2}a_0$, is summable and its only infinity can be at x_0. Therefore the derived series of the Fourier series of $G(x)$ converges (C1) at x_0 to $-\frac{1}{2}a_0$; hence the Fourier series of $f(x)$ converges (C1) at x_0 to 0.

For the case (*b*), $F'(x_0)$ exists if $m<k$ and then the Fourier series of $f(x)$ converges (C1) at x_0 to 0.

Similarly for the case (*c*) if $m<k<1$.

General result on the difference between the convergence properties.

6. Generally, the Fourier series † of $f(x)$ will converge (C1) at x_0 to 0 if

$$\chi(z) \prec \frac{1}{z},\ \psi(z) \succ \log\frac{1}{|z^2|},\ \text{and}\ \frac{\chi}{\psi} \prec z.$$

If, *at the same time*,

$$\chi(z) \succ z\sqrt{\psi''(z)},$$

then the Legendre series does *not* converge (C1) at x_0, and we have a difference between the convergence properties of the two series.

* See a paper of mine to be published soon.

† The question of the convergence (C1) of the Fourier series of a function at a point when the function has an *infinite* discontinuity of the second kind will be answered at some length in a subsequent communication.

4

On the steady motion of a viscous liquid due to the translation of a tore parallel to its axis.

BY

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1. The object of the present paper is to determine the steady motion set up in a viscous fluid at rest at infinity due to the translation of a tore, with small constant velocity parallel to its axis. The success of the solution depends on the determination of Stokes' stream function which has been obtained in (23). The force necessary to maintain the motion of the tore has been calculated from the stream function.

2. If (ρ, z) be the cylindrical coordinates, then the corresponding components of the velocity of the fluid (u, v) are expressed in terms of Stokes' stream function by

$$u=\frac{1}{\rho}\frac{\partial\psi}{\partial z}, \quad v=-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} \qquad \dots \quad (1)$$

In the case of steady motion in a viscous liquid, ψ satisfies the equation*

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]^2\psi=0 \qquad \dots \quad (2)$$

If the solid be moving with constant velocity V in the positive direction of the z-axis, the boundary conditions to be satisfied by ψ, on the assumption of no slipping at the surface of the solid, are

$$\frac{\partial\psi}{\partial z}=0 \text{ and } \frac{\partial\psi}{\partial\rho}=-\rho V \qquad \dots \quad (3)$$

* Lamb. Hydrodynamics. (4th edition) p. 592.

The equation (3) can be replaced by

$$\psi+\tfrac{1}{2}\rho^2 V=0, \quad \frac{\partial}{\partial n}(\psi+\tfrac{1}{2}\rho^2 V)=0 \qquad \ldots \quad (4)$$

where n denotes the outward-drawn normal to the surface.

The total force exerted by the fluid on the solid has been shown by Stimson and Jeffery* to be given by

$$F=k\pi\int \rho^3 \frac{\partial}{\partial n}\left[\frac{1}{\rho^2}\left(\frac{\partial^2}{\partial \rho^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{\rho}\frac{\partial}{\partial \rho}\right)\psi\right]ds \qquad \ldots \quad (5)$$

where k is the coefficient of viscosity; the integral is to be taken round a meridian section of the solid.

3. If we use curvilinear coordinates (α, β) in the meridian plane given by

$$\alpha+i\beta=\log\frac{\rho+a+iz}{\rho-a+iz} \qquad \ldots \quad (6)$$

we have

$$\rho=\frac{a\ \sinh\alpha}{\cosh\alpha-\cos\beta}, \quad z=\frac{-a\ \sin\beta}{\cosh\alpha-\cos\beta} \qquad \ldots \quad (7)$$

and

$$h^2=\left(\frac{\partial\alpha}{\partial\rho}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2=\frac{(\cosh\alpha-\cos\beta)^2}{a^2}$$

so that

$$h=\frac{\cosh\alpha-\cos\beta}{a} \qquad ()8$$

The curves α=constant and β=constant are two systems of orthogonal coaxial circles in the plane of (ρ, z), the distance between the limiting points being $2a$. The surfaces obtained by rotating the curves α=constant about the axis of z are a system of tores.

* "The motion of two spheres in a viscous liquid", Proc. Roy. Soc., Vol. 111, Ser. A, p. 110.

4. Let us first start with the equation

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]\psi=0 \qquad \dots \quad (9)$$

If $\rho+iz=f(\alpha+i\beta)$, the equation (9) can be transformed into

$$\left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}-\frac{1}{\rho}\frac{\partial\rho}{\partial\alpha}\cdot\frac{\partial}{\partial\alpha}-\frac{1}{\rho}\frac{\partial\rho}{\partial\beta}\frac{\partial}{\partial\beta}\right]\psi=0 \qquad \dots \quad (10)$$

If we now put $\quad \psi=\sqrt{\rho}\,\phi$

and make use of the relations

$$\frac{\partial^2\rho}{\partial\alpha^2}+\frac{\partial^2\rho}{\partial\beta^2}=0$$

and $$\left(\frac{\partial\rho}{\partial\alpha}\right)^2+\left(\frac{\partial\rho}{\partial\beta}\right)^2=\frac{1}{h^2}$$

we find that ϕ satisfies the equation

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2}-\frac{3\phi}{4h^2\rho^2}=0 \qquad \dots \quad (11)$$

In our co-ordinates, this equation takes the form

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2}-\frac{3\phi}{4\sinh^2\alpha}=0 \qquad \dots \quad (12)$$

To obtain a solution of (12), let us assume that

$$\phi=U\cos(n\beta+\epsilon) \qquad \dots \quad (13)$$

where U is a function of α alone. Then U satisfies the equation

$$\frac{d^2U}{d\alpha^2}-n^2U-\frac{3U}{4\sinh^2\alpha}=0 \qquad \dots \quad (14)$$

Put $U = \Pi/\sqrt{\sinh\alpha}$,

then Π satisfies the equation

$$(1-\mu^2)\frac{d^2\Pi}{d\mu^2}+(n-\tfrac{1}{2})(n+\tfrac{1}{2})\Pi=0 \qquad \ldots \quad (15)$$

where $\mu=\cosh\alpha$.

A solution of (15) can be written as

$$\Pi=\int P_{n-\frac{1}{2}}(\mu)\,d\mu$$

which, with the help of the recurrence formula

$$P'_{n+\frac{1}{2}}(\mu)-P'_{n-\frac{1}{2}}(\mu)=2n\,P_{n-\frac{1}{2}}(\mu)$$

reduces to

$$\Pi=\frac{1}{2n}[P_{n+\frac{1}{2}}(\mu)-P_{n-\frac{1}{2}}(\mu)] \qquad \ldots \quad (16)$$

except when $n=0$. In the latter case, with the help of the formula

$$P'_{\frac{1}{2}}(\mu)-\mu P'_{-\frac{1}{2}}(\mu)=\tfrac{1}{2}P_{-\frac{1}{2}}(\mu)$$

we have

$$\Pi=2[\mu P_{-\frac{1}{2}}(\mu)-P_{\frac{1}{2}}(\mu)] \qquad \ldots \quad (16a)$$

Therefore we have, as a solution of (12)

$$\phi=A_0(\mu P_{-\frac{1}{2}}-P_{\frac{1}{2}})+\sum_{n=1}^{n=\infty}A_n\,(P_{n-\frac{1}{2}}-P_{n+\frac{1}{2}})\cos(n\beta+\epsilon) \quad \ldots \quad (17)$$

Similarly we can prove that

$$\phi=B_0(\mu Q_{-\frac{1}{2}}-Q_{\frac{1}{2}})+\sum_{n=1}^{n=\infty}B_n\,(Q_{n-\frac{1}{2}}-Q_{n+\frac{1}{2}})\cos(n\beta+\epsilon) \quad \ldots \quad (17a)$$

is also a solution of (12).

The complete solution of (9) can now be written in the form

$$\psi=(\mu-\cos\beta)^{-\frac{1}{2}}[\{A_0\,(\mu P_{-\frac{1}{2}}-P_{\frac{1}{2}})+B_0(\mu Q_{-\frac{1}{2}}-Q_{\frac{1}{2}})\}$$

$$+\sum_{n=1}^{n=\infty}\{A_n\,(P_{n-\frac{3}{2}}-P_{n+\frac{1}{2}})+B_n\,(Q_{n-\frac{3}{2}}-Q_{n+\frac{1}{2}})\}\cos\,(n\beta+\epsilon)] \quad \ldots \quad (18)$$

5. Let us now try to solve the equation (2). Since the operator

$$\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}$$

is linear in z, we can easily show that

$$\psi=\psi_1+z\frac{\partial\psi_2}{\partial z}$$

is a solution of (2), if ψ_1 and ψ_2 are any two solutions of (9). Again, since $\frac{\partial\psi_2}{\partial z}$ is also a solution of (9), we can replace the above expression for ψ by

$$\psi=\psi_1+z\psi_2 \quad (19)$$

Substituting the values of ψ_1 and ψ_2 from (18) in (19) and only keeping those terms which make the velocity zero at infinity, we have

$$\psi=(\mu-\cos\beta)^{-\frac{1}{2}}\sum_{n=0}^{n=\infty}(a_n\cos n\beta+b_n\sin n\beta)V_n$$

$$+\sin\beta(\mu-\cos\beta)^{-\frac{3}{2}}\sum_{n=0}^{n=\infty}(c_n\cos n\beta+d_n\sin n\beta)V_n \quad \ldots \quad (20)$$

where $\qquad V_0=\mu P_{-\frac{1}{2}}-P_{\frac{1}{2}}$

and $\qquad V_n=P_{n-\frac{3}{2}}-P_{n+\frac{1}{2}}, \quad$ for $n\geq 1$

With the help of the recurrence relations

$$\left.\begin{aligned} \mu V_1 &= V_0 + \frac{5}{8} V_2 \\ \text{and} \qquad \mu V_n &= \frac{2n-3}{4(n-1)} V_{n-1} + \frac{2n+3}{4(n+1)} V_{n+1} \\ &\qquad \text{for } n \geq 2 \end{aligned}\right\} \qquad (21)$$

the equation (20) can be written as

$$\psi = (\mu - \cos\beta)^{-\frac{3}{2}} [(A_0 \mu V_0 + B_0 V_1)$$

$$+ \sum_{n=1}^{n=\infty} \{(A_n V_{n-1} + B_n V_{n+1}) \cos n\beta +$$

$$(C_n V_{n-1} + D_n V_{n+1}) \sin n\beta\}] \qquad \ldots \quad (22)$$

6. In the case of the steady motion of a tore in a viscous fluid parallel to its axis, let us assume

$$\psi = (\mu - \cos\beta)^{-\frac{3}{2}} \chi \qquad \ldots \quad (23)$$

where

$$\chi = (A_0 \mu V_0 + B_0 V_1) + \sum_{n=1}^{n=\infty} (A_n V_{n-1} + B_n V_{n+1}) \cos n\beta \ldots \quad (24)$$

The boundary conditions (4) become

$$\chi = -\tfrac{1}{2} \frac{a^2 V(\mu^2 - 1)}{(\mu - \cos\beta)^{\frac{3}{2}}}$$

$$= -\tfrac{1}{2} a^2 V(\mu^2 - 1) \sum_{n=0}^{n=\infty} L_n \cos n\beta \cdots \quad (25a)$$

and
$$\frac{d\chi}{d\mu}=\frac{1}{4}\cdot\frac{a^2V(\mu^2-1)}{(\mu-\cos\beta)^{\frac{3}{2}}}-\frac{a^2V\mu}{(\mu-\cos\beta)^{\frac{1}{2}}}$$

$$=\frac{1}{4}a^2V(\mu^2-1)\sum_{n=0}^{n=\infty}M_n\cos n\beta-a^2V\mu\sum_{n=0}^{n=\infty}L_n\cos n\beta \quad \ldots (25b)$$

where
$$\left.\begin{aligned}L_0&=\frac{1}{\pi}\int_0^{\pi}\frac{d\beta}{(\mu-\cos\beta)^{\frac{1}{2}}}=\frac{\sqrt{2}}{\pi}Q_0(\mu)\\ L_n&=\frac{2}{\pi}\int_0^{\pi}\frac{\cos n\beta d\beta}{(\mu-\cos\beta)^{\frac{1}{2}}}=\frac{2\sqrt{2}}{\pi}Q_n(\mu)\end{aligned}\right\} \quad \ldots (26a)$$

$$\left.\begin{aligned}M_0&=\frac{1}{\pi}\int_0^{\pi}\frac{d\beta}{(\mu-\cos\beta)^{\frac{3}{2}}}=-\frac{2\sqrt{2}}{\pi}\frac{dQ_0}{d\mu}\\ M_n&=\frac{2}{\pi}\int_0^{\pi}\frac{\cos n\beta d\beta}{(\mu-\cos\beta)^{\frac{3}{2}}}\\ &=-\frac{4\sqrt{2}}{\pi}\frac{dQ_n}{d\mu}\end{aligned}\right\} \quad \ldots (26b)$$

If the boundary be given by $\mu=\mu_0$ $(a=a_0)$ then at $\mu=\mu_0$

$$\left.\begin{aligned}&A_0\mu V_0+B_0V_1=-\tfrac{1}{2}a^2V(\mu^2-1)L_0\\ &A_0V_0+A_0\mu\frac{dV_0}{d\mu}+B_0\frac{dV_1}{d\mu}=\frac{1}{4}a^2V(\mu^2-1)M_0 \quad a^2V\mu L_0\end{aligned}\right\} \quad (27)$$

and for $n\geq 1$

$$\left.\begin{aligned}&A_nV_{n-1}+B_nV_{n+1}=-\frac{1}{2}a^2V(\mu^2-1)L_n\\ &A_n\frac{dV_{n-1}}{d\mu}+B_n\frac{dV_{n+1}}{d\mu}=\frac{1}{4}a^2V(\mu^2-1)M_n-a^2V\mu L_n\end{aligned}\right\} \quad \ldots (28)$$

Simplifying with the help of the relations

$$\frac{dV_0}{d\mu}=\frac{1}{2}P_{-\frac{1}{2}}$$

and
$$\frac{dV_n}{d\mu}=-2n\,P_{n-\frac{1}{2}},\qquad \text{for } n\geq 1$$

we get

$$\left.\begin{aligned}
&A_0\left[V_0V_1+2\mu V_0P_{\frac{1}{2}}+\frac{1}{2}\mu V_1P_{-\frac{1}{2}}\right]\\
&=a^2V\left[-(\mu^2-1)L_0P_{\frac{1}{2}}+\frac{1}{4}(\mu^2-1)M_0V_1-\mu L_0V_1\right]\\
&B_0\left[V_0V_1+2\mu V_0P_{\frac{1}{2}}+\frac{1}{2}\mu V_1P_{-\frac{1}{2}}\right]\\
&=-a^2V\left[\frac{1}{2}(\mu^2-1)L_0\left\{V_0+\frac{1}{2}\mu P_{-\frac{1}{2}}\right\}+\frac{1}{4}(\mu^2-1)\mu M_0V_0\right.\\
&\qquad\qquad\left.-\mu^2L_0V_0\right]
\end{aligned}\right\}\tag{29}$$

and for $n\geq 1$

$$\left.\begin{aligned}
&A_n\left[(n+1)V_{n-1}P_{n+\frac{1}{2}}-(n-1)V_{n+1}P_{n-\frac{3}{2}}\right]\\
&\qquad=\frac{1}{2}a^2V\left[-(n+1)(\mu^2-1)L_nP_{n+\frac{1}{2}}\right.\\
&\qquad\left.+\frac{1}{4}(\mu^2-1)M_nV_{n+1}-\mu L_nV_{n+1}\right]\\
&B_n\left[(n+1)V_{n-1}P_{n+\frac{1}{2}}-(n-1)V_{n+1}P_{n-\frac{3}{2}}\right]\\
&\qquad=\frac{1}{2}a^2V\left[(n-1)(\mu^2-1)L_nP_{n-\frac{3}{2}}\right.\\
&\qquad\left.-\frac{1}{4}(\mu^2-1)M_nV_{n-1}+\mu L_nV_{n-1}\right]
\end{aligned}\right\}\tag{30}$$

μ_0 being put for μ in the above expressions for A_0, B_0, A_n, B_n.

Thus the stream function is completely determined.

7. In order to calculate the forces necessary to maintain the motion of the tore, we substitute the value of ψ in (5) and find, after some calculation, that

$$F=-\frac{2\pi^2 k}{a}\Big[A_0L_0\Big\{\frac{3}{2}\sinh^3 a\, V_0-\mu\Big(2\mu V_0-\frac{1}{2}\sinh^3 a\, P_{-\frac{1}{2}}\Big)\Big\}$$

$$-\frac{1}{2}A_0M_0\sinh^3 a\Big(\sinh^3 aP_{-\frac{1}{2}}+6\mu V_0\Big)$$

$$-\frac{1}{2}A_0M_0\mu\Big\{3\sinh^3 a\cosh aP_{-\frac{1}{2}}-\Big(12\cosh^2 a-2\sinh^2 a\Big)V_0\Big\}$$

$$+\frac{1}{2}B_0M_0\Big\{(12\cosh^2 a-3\sinh^2 a)V_1+12\sinh^3 a\cosh a\, P_{\frac{1}{2}}\Big\}$$

$$+A_0N_0\sinh^3 a\Big\{\frac{3}{2}\sinh^3 a\cosh aP_{-\frac{1}{2}}+3\Big(\sinh^2 a-\frac{3}{2}\cosh^2 a\Big)V_0\Big\}$$

$$-B_0N_0\sinh^3 a\Big\{6\sinh^3 a\, P_{\frac{1}{2}}+\frac{9}{2}\cosh a\, V_1\Big\}\Big]$$

$$-\frac{\pi^4 k}{a}\sum_{n=1}^{n=\infty}\Big[4n(2n-1)A_nL_n\Big\{(n-1)\sinh^3 aP_{n-\frac{1}{2}}+\cosh a\, V_{n-1}\Big\}$$

$$+4n(2n+1)B_nL_n\Big\{(n+1)\sinh^3 a\, P_{n+\frac{1}{2}}+\cosh a\, V_{n+1}\Big\}$$

$$+\frac{1}{2}A_nM_n\Big\{12(n-1)\sinh^3 a\cosh aP_{n-\frac{1}{2}}$$

$$+\Big(2n-3\Big)\sinh^3 a\, V_{n-1}+12\cosh^2 aV_{n-1}\Big\}$$

$$+\frac{1}{2}B_nM_n\left\{12(n+1)\sinh^3\alpha\cosh\alpha P_{n+\frac{1}{2}}\right.$$

$$\left.-(2n+3)\sinh^2\alpha\, V_{n+1}+12\cosh^2\alpha\, V_{n+1}\right\}$$

$$-3A_nN_n\sinh^3\alpha\left\{2(n-1)\sinh^2\alpha P_{n-\frac{1}{2}}+\frac{3}{2}\cosh\alpha V_{n-1}\right\}$$

$$-3B_nN_n\sinh^2\alpha\left\{2(n+1)\sinh^2\alpha\; P_{n+\frac{1}{2}}+\frac{3}{2}\cosh\alpha V_{n+1}\right\}\Bigg] \quad (31)$$

where L_0, M_0, L_n, M_n are given by (26*a*), (26*b*) and

$$N_0=\frac{1}{\pi}\int_0^\pi\frac{d\beta}{(\mu-\cos\beta)^{\frac{5}{2}}}=\frac{4\sqrt{2}}{3\pi}\cdot\frac{d^2Q_0}{d\mu^2}$$

$$N_n=\frac{2}{\pi}\int_0^\pi\frac{\cos n\beta d\beta}{(\mu-\cos\beta)^{\frac{5}{2}}}=\frac{8\sqrt{2}}{3\pi}\,\frac{d^2Q_n}{d\mu^2}$$

In the above expression, α is to be replaced by α_0.

The numerical calculations for F are, however, rendered extremely difficult by the appearance of the functions P_n of fractional order in the solution.

In conclusion, I wish to express my thanks to Dr. N. R. Sen for guidance in course of the work.

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ON KIEPERT'S SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE

BY

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1. In a paper entitled "Auflösung der Gleichungen fünften Grades" published in *Crelle's Journal*, Vol. 87, Kiepert gave all the formulæ necessary for the solution of the general equation of the fifth degree; but he did not illustrate his method by means of any numerical example. As a matter of fact, there was a difficulty in his doing so, because he retained in the set of his formulæ some ambiguities of sign unexplained. In the year 1907, Morgenstern* noticed this imperfection in Kiepert's method and with a view to throwing light on this obscure point he started with some numerical equations and by carefully studying the nature of their solutions, he was led to formulate in a definite way the method of Kiepert. The object of the present paper is to study the same question by a method which serves to explain the ambiguities in a clear and convincing manner. I have been careful to give an idea of Kiepert's method to avoid abruptness in the treatment. It is believed that my treatment of the defect in Kiepert's method is an improvement on Morgenstern's treatment.

2. The *first* step in Kiepert's method consists in reducing the general quintic equation

$$x^5+Ax^4+Bx^3+Cx^2+Dx+E=0$$

to the form

$$z^5+5lz^2-5mz+n=0$$

by means of the transformation

$$z=x^2-ux+v$$

where u and v have to be chosen in a suitable manner. For this purpose, I eliminate x between the equations

$$x^5+Ax^4+Bx^3+Cx^2+Dx+E=0 \text{ and } x^2-ux+v-z=0$$

* *Beitrage zur Numerischen Lösung der Gleichungen Funften Grades* (Halle, 1907).

and write the eliminant directly as

$$(x_1^5+Ax_1^4+Bx_1^3+Cx_1^2+Dx_1+E)(x_2^5+Ax_2^4+Bx_2^3+Cx_2^2+Dx_2+E)=0$$

where x_1+x_2 and x_1x_2 are to be replaced by u and $v-z$ respectively. Performing this operation, the eliminant appears in the form

$$\begin{aligned}&(v-z)^5+(v-z)^4(Au-2B+A^2)+(v-z)^3\{Bu^2+(AB-3C)u+2D-2AC+B^2\}\\&+(v-z)^2\{Cu^3+(AC-4D)u^2+(5E-3AD+BC)u+C^2+2AE-2BD\}\\&+(v-z)\{Du^4+(AD-5E)u^3+(BD-4AE)u^2+(CD-3BE)u+D^2-2CE\}\\&+E(u^5+Au^4+Bu^3+Cu^2+Du+E)=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)\end{aligned}$$

Let u and v be so chosen that the co-efficients of z^4 and z^3 in (1) are each equal to zero. This is satisfied if u and v are such that

$$\left.\begin{aligned}&(2A^2-5B)u^2+(4A^3-13AB+15C)u+(2A^4-8A^2B+10AC\\&+3B^2-10D)=0\\&\text{and } 5v=-Au-A^2+2B\end{aligned}\right\}\ldots \quad (2)$$

Putting now

$$\left.\begin{aligned}5l&=-C(u^3+Au^2+Bu+C)+D(4u^2+3Au+2B)-E(5u+2A)\\&\quad-10v^3\\5m&=-D(u^4+Au^3+Bu^2+Cu+D)+E(5u^3+4Au^2+3Bu+2C)\\&\quad+5v^4+10lv\\n&=-E(u^5+Au^4+Bu^3+Cu^2+Du+E)-v^5-5lv^2+5mv,\end{aligned}\right\}\ldots(3)$$

the equation (1) may be written as

$$z^5+5lz^2-5mz+n=0 \quad \ldots \quad (4)$$

3. Following Kiepert, I *next* take the quintic

$$\Delta^3y^5+10\Delta^2y^3+45\Delta y-216g_3=0 \quad \ldots \quad (5)$$

where Δ stands for $g_2^3-27g_3^2$, g_2 and g_3 being two arbitrary parameters.

Putting $z=-\dfrac{\alpha+\beta y}{3+\Delta y^2}$, it is found, by a similar method, that the equation (5) is transformed to

$$z^5+5Lz^2-5Mz+N=0 \quad \ldots \quad (6)$$

where

$$\left.\begin{aligned}1728\,g_2^3\Delta L&=8\Delta^2\alpha^3-72\Delta\alpha\beta^2+216\,g_3\,(\Delta\alpha^2\beta-\beta^3)\\1728\,g_2^3\Delta M&=\Delta^2\alpha^4+18\Delta\alpha^2\beta^2-27\beta^4+216\,g_3\beta^3\alpha\\1728\,g_2^3\Delta^2N&=\Delta^3\alpha^5+10\Delta^2\alpha^3\beta^2+45\Delta\alpha\beta^4+216\,g_3\beta^5\end{aligned}\right\}\ldots \quad (7)$$

As the equations (4) and (6) have the same form, it follows that the former admits of being transformed to the equation (5) by means of the transformation

$$z = -\frac{\alpha+\beta y,}{3+\Delta y^2}, \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

if α, β, g_2, g_3 can be so chosen as to satisfy the equations

$$\left.\begin{aligned} 1728 g_2{}^3 \Delta l &= 8\Delta^2\alpha^3 - 72\Delta\alpha\beta^2 + 216 g_3(\Delta\alpha^2\beta - \beta^3), \\ 1728 g_2{}^3 \Delta m &= \Delta^2\alpha^4 + 18\Delta\alpha^2\beta^2 - 27\beta^4 + 216 g_3\beta^3\alpha, \\ 1728 g_2{}^3 \Delta^2 n &= \Delta^3\alpha^5 + 10\Delta^2\alpha^3\beta^2 + 45\Delta\alpha\beta^4 + 216 g_3\beta^5. \end{aligned}\right\} \quad \ldots \quad (9)$$

Thus the solution of the general quintic is made to depend on that of the equation (5), the well-known Brioschi's resolvent.

4. I proceed now to the reduction of the equations (9).
Multiply the second by $\Delta\alpha$ and subtract the last from this: then

$$1728 g_2{}^3 \Delta^2(\alpha m - n) = 8\Delta^2\alpha^3\beta^2 - 72\Delta\alpha\beta^4 + 216 g_3\beta^2(\Delta\alpha^2\beta - \beta^3)$$

$$= 1728 g_2{}^3 \Delta l\beta^2.$$

Therefore $(\alpha m - n)\Delta = l\beta^2$ ... ... (10)

Eliminate g_3 between the last two equations: then

$$1728\, g_2^3\ \Delta\,(\Delta n\alpha - m\beta^2) = (\Delta\alpha^2 + 3\beta^2)^3,$$

which by means of (10) becomes

$$1728\ g_2^3\ l^2(nl\alpha - m^2\alpha + mn) = \Delta(\alpha^2 l + 3\alpha m - 3n)^3 \quad \ldots \quad (11)$$

Using the equation (10), the first two equations may be written:

$$1728\ g_2^3 l^2 = 8l\Delta\alpha^3 - 72\Delta\alpha(\alpha m - n) + 216\ g_3\beta(\alpha^2 l - \alpha m + n),$$

$$1728\ g_2^3 ml^2 = \Delta l^2\alpha^4 + 18\Delta\alpha^2 l(\alpha m - n) - 27\Delta(\alpha m - n)^2$$
$$+ 216 g_3\beta\alpha l(\alpha m - n).$$

g_2^3 be now eliminated, then

$$216\ g_3\beta(nl\alpha - m^2\alpha + mn)$$
$$= \Delta(l^2\alpha^4 + 10\alpha^3 lm - 18\alpha^2 ln + 45\alpha^2 m^2 - 18\alpha mn - 27n^2) \quad \ldots \quad (12)$$

Thus the equations (9) may be replaced by the equations (10), (11) and (12).

Put then

$$a^2 l + 3am - 3n = p.$$

$$nla - m^2 a + mn = q$$

and $$l^2 a^4 + 10a^3 lm - 18a^2 ln + 45a^2 m^2 - 18amn - 27n^2 = r.$$

From the relation $g_2^3 = \Delta + 27\, g_3^2$, (11) may be written :

$$1728 \times 27\ g_3^2 l^2 q = \Delta(p^3 - 1728 l^3 q) \qquad \dots \quad (13)$$

Squaring (12) we have also

$$216 \times 216\ g_3^2 \beta^2 q^2 = \Delta^2 r^2 \qquad \dots \quad (14)$$

From (13) and (14),

$$\frac{\beta^2 q}{l^2} = \frac{\Delta r^2}{p^3 - 1728 l^3 q},$$

which by (10) becomes

$$\frac{q(am-n)}{l^3} = \frac{r^2}{p^3 - 1728 l^3 q}.$$

By a somewhat laborious calculation the above may be simplified to

$$p^3(\lambda a^2 + \mu a + \nu) = 0, \qquad \dots \quad \dots \quad (15)$$

where λ, μ, ν denote respectively

$$l^4 - lmn + m^3,$$

$$11 l^3 m + l n^2 - 2m^2 n,$$

and $$-27 l^3 n + 64 l^2 m^2 + mn^2.$$

5. Let us now discuss the process through which the equation (4) is transformed to the equation (5). For this purpose, we shall consider the equations (10), (11), (12) and (15).

Calculate a from the equation (15),

$$i.e., \quad \lambda a^2+\mu a+\nu=0. \quad \ldots \quad \ldots \quad (15)'$$

Take any *one* of the values of a and find the values of p, q, r corresponding to it. From the equations (10), (11) and (12), $\frac{\beta^2}{\Delta}$, $\frac{g_2^3}{\Delta}$ and $\frac{g_3\beta}{\Delta}$ are all known quantities. Represent them by a, b, c, respectively, then we have

$$\left.\begin{aligned} \Delta&=\frac{\beta^2}{a},\\ g_3&=\frac{\beta c}{a},\\ g_2^3&=\frac{\beta^2 b}{a}, \end{aligned}\right\} \quad \ldots \quad (16)$$

so that β remains to be chosen arbitrarily. In particular, let $\beta=a$. then the equation (4) is transformed to the equation

$$a^3y^6+10a^2y^3+45ay-216c=0 \quad \ldots \quad (17)$$

by means of the transformation

$$z=-\frac{a+ay}{3+ay^2}.$$

The equation (17) is different for a different choice of β, but *it is easy to see that in whatever way β may be chosen the roots of the equation* (4), *as determined from the value of* z, *are not affected.* A convenient choice of β is, therefore, sufficient for our purpose.

If, however, we agree to maintain the signs of Δ and g_3 invariable in the equation (5) then the sign of β will depend upon the sign of c. For, from (16) we have

$$\frac{\Delta}{g_3}=\frac{\beta}{c}.$$

6. It will now be interesting to obtain those formulae which Kiepert adopts.

Assign to Δ a value $\epsilon l^2 q$ where ϵ is either $+1$ or -1; then

β^2 should be $\epsilon a l^2 q$ which by means of $(15)'$ readily reduces to Kiepert's form,

$$g_2^3 = \epsilon l^2 q b = \epsilon \frac{p^3}{1728}, \text{ whence } 12\, g_2 = \epsilon p,$$

$$g_3 = \frac{\beta c}{a} = \frac{\epsilon l^2 q c}{\beta}.$$

From what has been remarked before there is no difficulty with regard to the choice of β: Kiepert omits the last formula which determines the particular sign of β to be chosen. Morgenstern includes this but omits the first as comparatively unimportant.

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